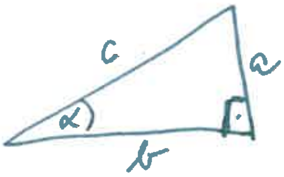
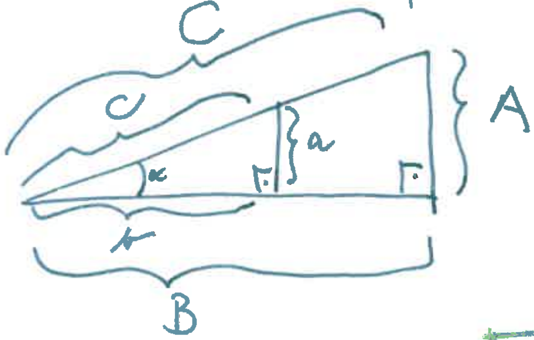
K čemu je dobrý sinus? (ŠKOMAM, 28.1.2019)

1 Co je sinus?

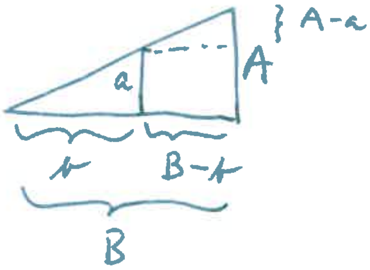


$$\boxed{\sin\alpha := \frac{a}{c}}$$

Aby věc měla smysl, je třeba ověřit, že $\boxed{\frac{a}{c} = \frac{A}{C}}$



Dk. a) Nejdříve dokážeme, že $\boxed{\frac{a}{b} = \frac{A}{B}}$



$$\frac{AB}{2} = \frac{ab}{2} + \frac{(A-a)(B-b)}{2} + (B-b)a$$

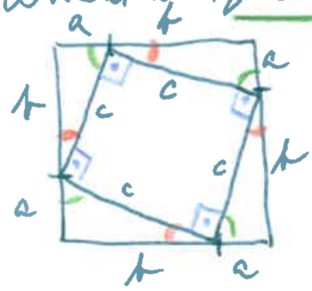
$$\Updownarrow$$

$$\cancel{AB} = \cancel{ab} + \cancel{AB} - aB - bA + \cancel{ab} + 2Ba - \cancel{2ab}$$

$$\Updownarrow$$

$$bA = aB \Rightarrow \frac{a}{b} = \frac{A}{B}$$

b) Dokážeme Pythagorovu větu: $\boxed{a^2 + b^2 = c^2}$



$$(a+b)^2 = c^2 + 4\,\frac{ab}{2}$$

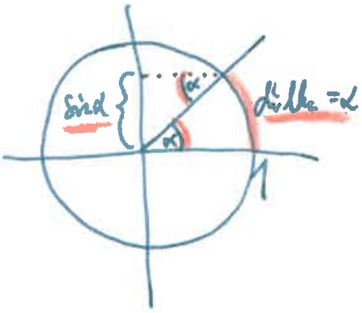
$$a^2 + \cancel{2ab} + b^2 = c^2 + \cancel{2ab}$$

c) $$\frac{a}{c} = \frac{A}{C} \Leftrightarrow \frac{a^2}{c^2} = \frac{A^2}{C^2} \Leftrightarrow \frac{a^2}{a^2+b^2} = \frac{A^2}{A^2+B^2} \Leftrightarrow$$

$$\Leftrightarrow \frac{\frac{a^2}{b^2}}{\frac{a^2}{b^2}+1} = \frac{\frac{A^2}{B^2}}{\frac{A^2}{B^2}+1}$$

viz a)

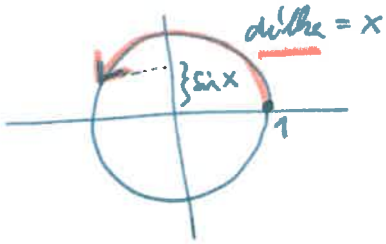
Cbd



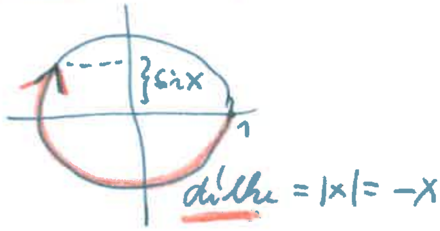
$\alpha \in (0, \frac{\pi}{2})$

Rozšíření:

$\boxed{x \in \mathbb{R},\ x \geq 0}$



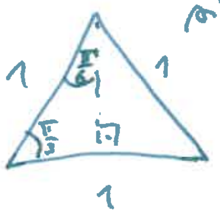
$\boxed{x \in \mathbb{R},\ x < 0}$



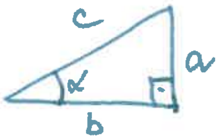
- $\sin x$ je definován pro všechna $x \in \mathbb{R}$,
- $\sin x \in \langle -1, 1 \rangle$ pro $\forall x \in \mathbb{R}$,
- sinus má periodu 2π,
- $\sin(x \pm \pi) = -\sin x$, ...

$\sin 0 = 0$, $\sin \frac{\pi}{2} = 1$

$\sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$, $\sin \frac{\pi}{6} = \frac{1}{2}$, $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$

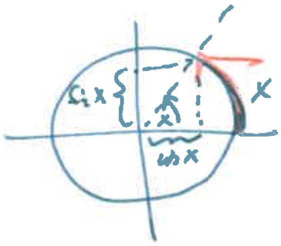


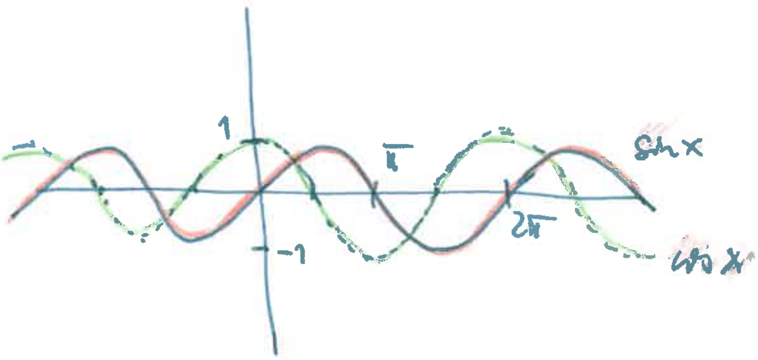
... podobně lze zavést kosinus



$\boxed{\cos \alpha = \frac{b}{c}} \left(= \sin\left(\frac{\pi}{2} - \alpha\right)\right)$

$\sin \alpha = \cos\left(\frac{\pi}{2} - \alpha\right) = \cos\left(\alpha - \frac{\pi}{2}\right)$





$\boxed{\forall x \in \mathbb{R} : \sin x = \cos\left(x - \frac{\pi}{2}\right)}$

... geometricky „jasné", ale: co je (přesně) sin 22 ?

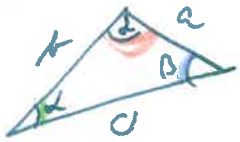
(... jak to kalkulačka počítá ?)

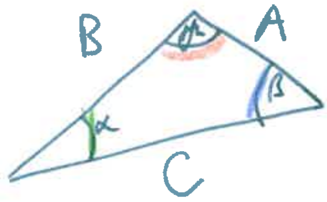
Slajdy

② K čemu je to dobré?

a) geometrické úlohy, trojúhelníky, ... $\longrightarrow$ SŠ matematika

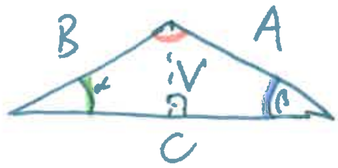
Př.:





$$\Longrightarrow \frac{a}{c} = \frac{A}{C}, \quad \frac{a}{b} = \frac{A}{B}, \quad \frac{b}{c} = \frac{B}{C}, \quad \dots$$

Dk.:

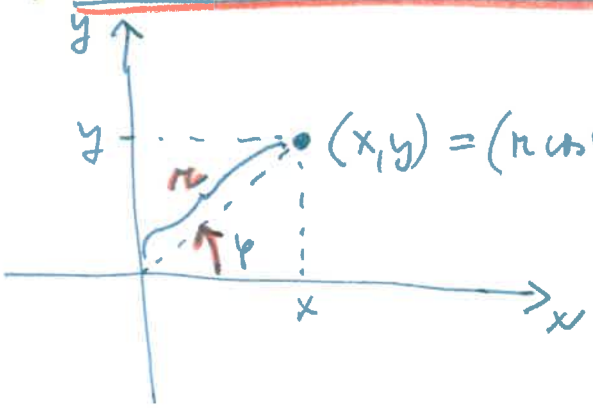


$$\left.\begin{array}{l} \sin\alpha = \dfrac{v}{b} = \dfrac{V}{B} \Rightarrow v = \dfrac{Vb}{B} \\[2ex] \sin\beta = \dfrac{v}{a} = \dfrac{V}{A} \Rightarrow v = \dfrac{Va}{A} \end{array}\right\} \Rightarrow \frac{b}{B} = \frac{a}{A} \Downarrow \frac{a}{b} = \frac{A}{B}$$

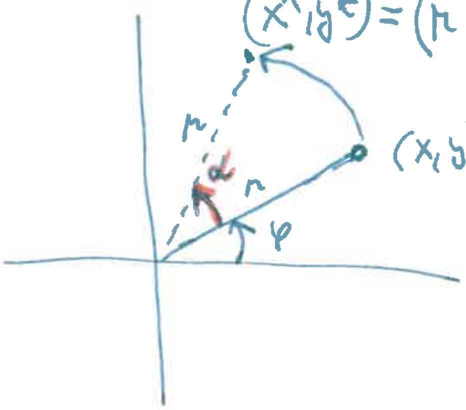
...

cbd.

b) polární souřadnice, rotace v rovině, $\mathbb{C}$



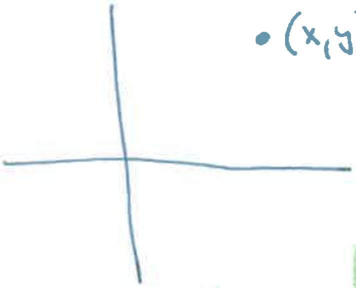
$$(x, y) = (r\cos\varphi, r\sin\varphi) \qquad \left(\begin{array}{l} r \geq 0 \\ \varphi \in \mathbb{R} \end{array}\right)$$



$$(x^*, y^*) = (r\cos(\varphi+\alpha), r\sin(\varphi+\alpha))$$

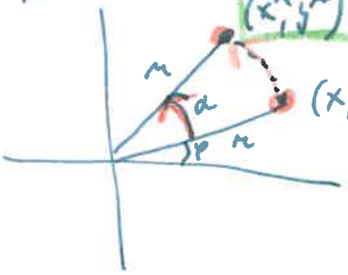
$$(x, y) = (r\cos\varphi, r\sin\varphi)$$

... rotace kolem počátku o úhel α



• $(x,y) \longleftrightarrow x+iy$ ... komplexní čísla

$x,y \in \mathbb{R},\ i^2=-1$



$$z^* = x^* + iy^*$$

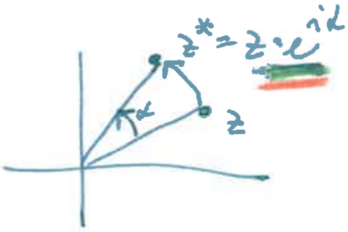
$$(x^*,y^*) \longleftrightarrow r\left(\cos(\varphi+\alpha) + i\sin(\varphi+\alpha)\right) = re^{i(\varphi+\alpha)} = \underbrace{re^{i\varphi}}_{=(x+iy)} \cdot e^{i\alpha} = z\cdot e^{i\alpha}$$

$$(x,y) = (r\cos\varphi, r\sin\varphi) = r(\cos\varphi, \sin\varphi) \longleftrightarrow$$

$$\longleftrightarrow \underbrace{r\cos\varphi + ir\sin\varphi}_{z\,=\,x+iy} = r(\underbrace{\cos\varphi + i\sin\varphi}_{=:\,e^{i\varphi}}) = re^{i\varphi}$$

$$e^{i\pi}+1 = \cos\pi + i\sin\pi + 1 = -1 + i0 + 1 = 0$$

$$\boxed{e^{i\pi}+1=0}$$



$$\boxed{e^x} := 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \qquad \forall x \in \mathbb{C} = \{x+iy : x,y \in \mathbb{R}\}$$

$\varphi \in \mathbb{R}$

$$e^{i\varphi} = 1 + (i\varphi) + \frac{(i\varphi)^2}{2!} + \frac{(i\varphi)^3}{3!} + \frac{(i\varphi)^4}{4!} + \dots = 1 + i\varphi - \frac{\varphi^2}{2!} - i\frac{\varphi^3}{3!} + \frac{\varphi^4}{4!} + i\frac{\varphi^5}{5!} - \dots$$

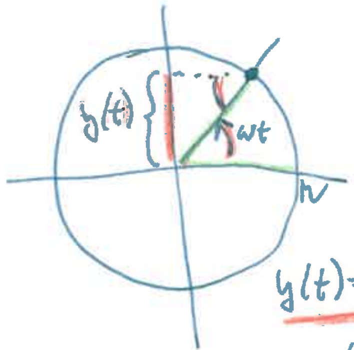
$$= \cos\varphi + i\sin\varphi$$

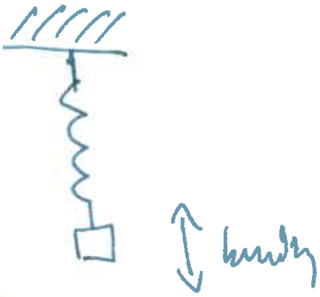
$\Downarrow$ (porovnáním reálných a imaginárních částí)

$$\boxed{\cos\varphi} = 1 - \frac{\varphi^2}{2!} + \frac{\varphi^4}{4!} - \frac{\varphi^6}{6!} + \dots$$

$$\boxed{\sin\varphi} = \varphi - \frac{\varphi^3}{3!} + \frac{\varphi^5}{5!} - \frac{\varphi^7}{7!} + \dots$$

c) harmonický pohyb





$y(t) = r\sin(\omega t)$

... rovnoměrný pohyb po kružnici (úhlová rychlost ω)

$$m y''(t) = m a = F = -k y(t)$$

$$y''(t) + \frac{k}{m} y(t) = 0$$

$$\Updownarrow$$

$$y(t) = C_1 \sin\left(\sqrt{\tfrac{k}{m}}\, t\right) + C_2 \cos\left(\sqrt{\tfrac{k}{m}}\, t\right)$$

$$y(0) = 0 \Rightarrow C_2 = 0$$

$$y(t) = C_1 \sin\left(\sqrt{\tfrac{k}{m}}\, t\right)$$

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d) Fourierovy řady

Slajdy