CO A K ČEMU JSOU DOBRÉ URČITÉ INTEGRÁLY

(ŠKOMAM – 27/1 2020)

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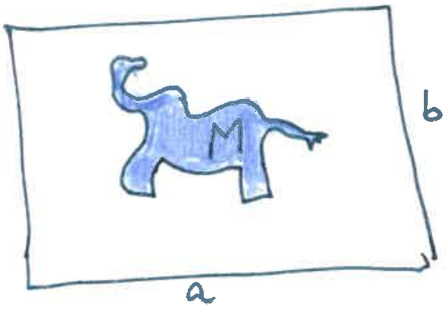
Motivace

① $M \subset \mathbb{R}^2$; „obsah" $M = ?$



... chemici
... Monte Carlo

② $M \subset \mathbb{R}^3$; „objem" $M = ?$

... pivo
... alkohol tester

③ „křivka" $k \subset \mathbb{R}^2$, „délka" $k = ?$

④ $M \subset \mathbb{R}^3$; „těžiště" $T = ?$

... příbor

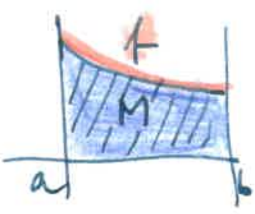
Jak definovat „..." ?
Jak spočítat ... ?

Speciální případ

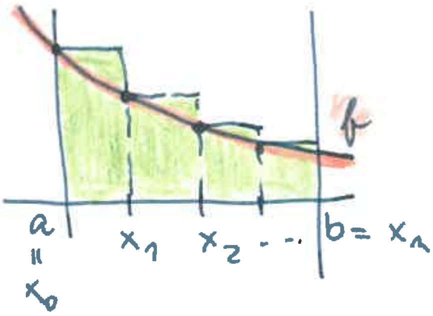
$f : \mathbb{R} \to \mathbb{R}$

f je • spojitá
• nezáporná na $\langle a,b\rangle$

$$M = \{(x,y) \in \mathbb{R}^2 : x \in \langle a,b\rangle \wedge 0 \le y \le f(x)\}$$



Co je obsah M ?



obsah $M \doteq f(x_0)\frac{b-a}{n} + f(x_1)\frac{b-a}{n} + \dots + f(x_{n-1})\frac{b-a}{n}$

$$= \frac{b-a}{n}\left[f(x_0) + f(x_1) + \dots + f(x_{n-1})\right]$$

$$\Big\downarrow n \to \infty$$

$$\int_a^b f(x)\,dx$$

Definice. Buď $f: \mathbb{R} \to \mathbb{R}$ spojitá na intervalu $\langle a, b\rangle$ (ne nutně nezáporná!)

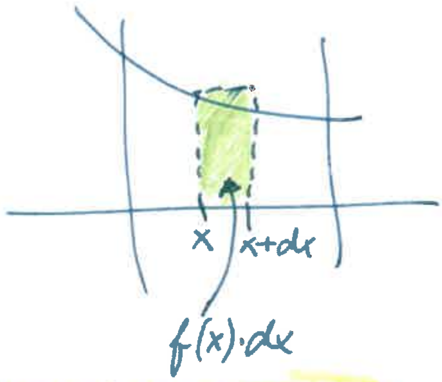
$$\boxed{\int_a^b f(x)\,dx \stackrel{\text{def.}}{=} \lim_{n\to\infty}\left[\frac{b-a}{n}\left(f(x_0) + \dots + f(x_{n-1})\right)\right],}$$

kde $x_i = a + i\,\frac{b-a}{n}$

Poznámka ke značení

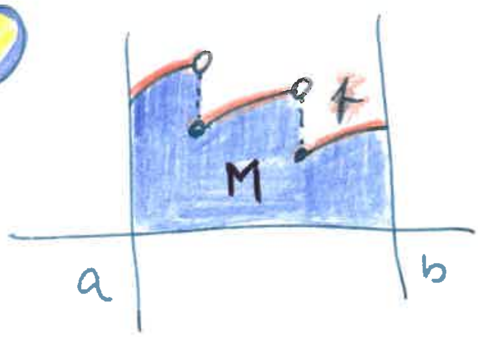
$$\int_a^b f(x)\,dx \approx \sum f(x)\,dx$$

součet



Možné zobecnění a s ním spojené označování

1



obsah $M = \int_a^b f(x)\,dx = \lim \dots$

2

$$\boxed{f(x) \stackrel{def.}{=} \begin{cases} 0, & x \in \mathbb{Q}, \\ 1, & x \in \mathbb{R}\setminus\mathbb{Q}. \end{cases}}$$ ... Dirichletova funkce

$$\text{"}\int_0^1 f(x)\,dx = \lim \dots\text{"} = 0,$$

$\forall\, 0 < r \in \mathbb{R}\setminus\mathbb{Q}$: $$\text{"}\int_r^{r+1} f(x)\,dx = \lim \dots\text{"} = 1,$$

$\forall\, 0 < r \in \mathbb{Q}$: $$\text{"}\int_0^{1+r} f(x)\,dx = \lim \dots\text{"} = 0$$

$\forall\, 0 < r \in \mathbb{R}\setminus\mathbb{Q}$: $$\text{"}\int_0^{1+r} f(x)\,dx = \lim \dots\text{"} = 1+r$$

$$\text{"}\int_0^{2020} f(x)\,dx\text{"} = 0 \qquad \text{"}\int_0^{\pi} f(x)\,dx\text{"} = \pi$$

... divné! ... !

Věta je-li f spojitá v $\langle a, b\rangle$, je
$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \left[\frac{b-a}{n} \left(f(\xi_1) + f(\xi_2) + \dots + f(\xi_n) \right) \right],$$
kde $\xi_i \in \langle x_{i-1}, x_i \rangle$ jsou zvoleny libovolně.
$(\forall n \in \mathbb{N})$

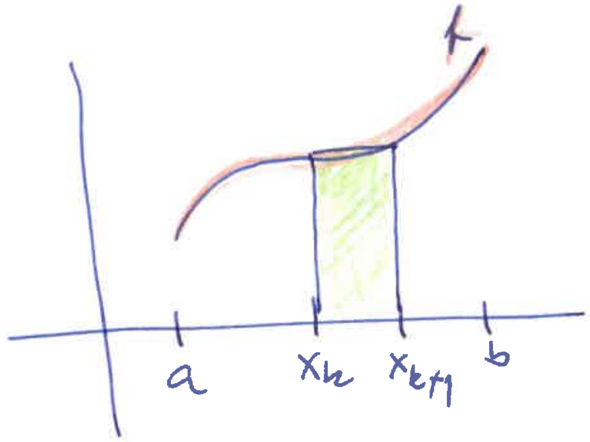
Definice
$$\boxed{\int_a^a f(x)\,dx \stackrel{def.}{=} 0} \qquad \boxed{\int_b^a f(x)\,dx \stackrel{def.}{=} -\int_a^b f(x)\,dx}$$
(f spojitá na $\langle a, b\rangle$)

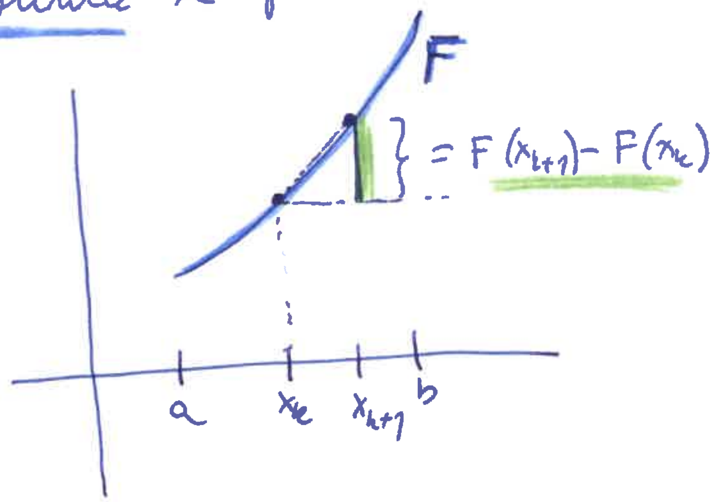
Pozorování

$0 \le f$ ... spojitá na $\langle a, b\rangle$

$F'(x) = f(x)$ pro každé $x \in \langle a, b\rangle$

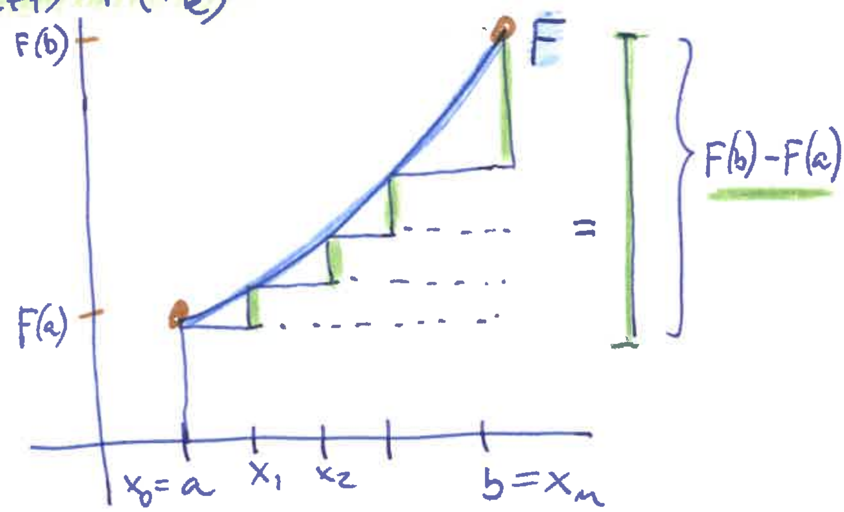
(F... primitivní funkce k f na $\langle a, b\rangle$)





$$f(x_k)(x_{k+1}-x_k) = F'(x_k)(x_{k+1}-x_k) \doteq \frac{F(x_{k+1})-F(x_k)}{x_{k+1}-x_k}\cdot(x_{k+1}-x_k) =$$

$$= F(x_{k+1}) - F(x_k)$$



$$\boxed{\int_a^b f(x)\,dx} \doteq \sum_{k=0}^{n-1} f(x_k)(x_{k+1}-x_k) \doteq \sum_{k=0}^{n-1}\left[F(x_{k+1})-F(x_k)\right] =$$

$$= \left(F(x_1)-F(x_0)\right) + \left(F(x_2)-F(x_1)\right) + \left(F(x_3)-F(x_2)\right) + \dots + \left(F(x_n)-F(x_{n-1})\right)$$

$$= F(x_n) - F(x_0) = \boxed{F(b)-F(a)}$$

Věta Nechť

- f a F jsou spojité na $\langle a,b\rangle$,
- $F'(x) = f(x)$ pro každé $x \in (a,b)$.

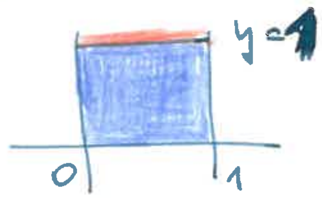
Pak platí

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a) \overset{\text{ozn.}}{=} \Big[F(x)\Big]_a^b}$$

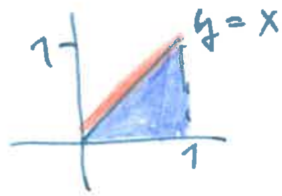
(Newtonova – Leibnizova formule)

Příklady

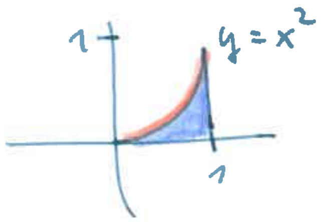
1. $\displaystyle\int_0^1 1\,dx = \Big[x\Big]_0^1 = 1 - 0 = 1$



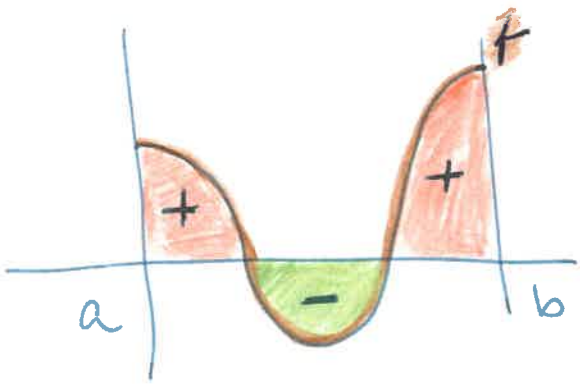
2. $\displaystyle\int_0^1 x\,dx = \Big[\frac{x^2}{2}\Big]_0^1 = \frac{1}{2} - 0 = \frac{1}{2}$



3. $\displaystyle\int_0^1 x^2\,dx = \Big[\frac{x^3}{3}\Big]_0^1 = \frac{1}{3} - 0 = \frac{1}{3}$



Pozorování – geometrický význam $\int_a^b f(x)\,dx$, kde f je spojitá na $\langle a,b\rangle$.

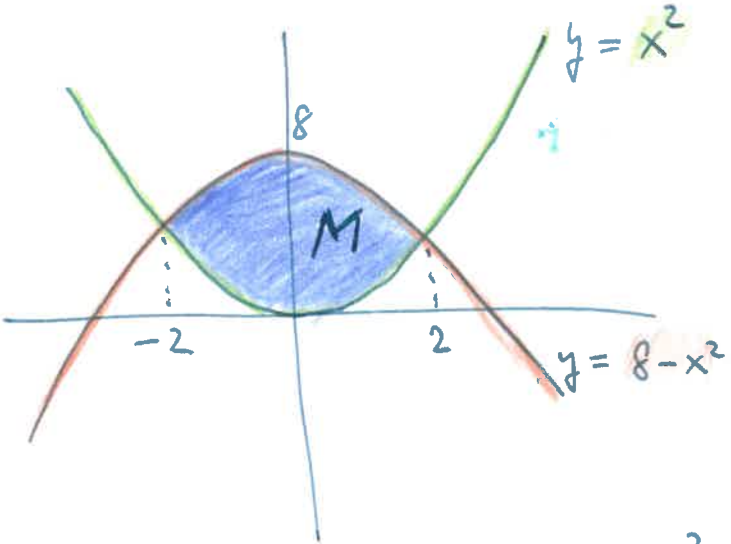


$$\int_a^b f(x)\,dx = \text{obsah} - \text{obsah}$$

Příklad.

Vypočtěte obsah množiny

$$M = \{(x,y) \in \mathbb{R}^2 :\ x^2 \le y \le 8 - x^2\}.$$



$$\underline{\text{obsah } M} = \int_{-2}^{2} 8 - x^2 \, dx - \int_{-2}^{2} x^2 \, dx =$$

$$= \left[8x - \frac{x^3}{3} \right]_{-2}^{2} - \left[\frac{x^3}{3} \right]_{-2}^{2} = \underline{\frac{64}{3}}$$

Úkol Buď

$$f(x) \overset{\text{def.}}{=} \begin{cases} 2x, & x \in \langle 0, 1 \rangle, \\ 4 - 2x, & x \in (1, \infty). \end{cases}$$

Vypočtěte

$$\int_{0}^{5} f(x)\, dx . \qquad (= -7)$$