ETH zürich

Od bitu ke qubitu a ještě kousek dál: Kvantové výpočty pro začátečníky

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Outline

1. Introduction
2. Single classical systems
3. Multiple classical systems
4. Quantum systems
5. Finale

Outline

Nultý slajd

Všiml si někdo pokusu o slovní hříčku v názvu?

ChatGPT chat "Czech Language Discussion":
Tento název, "Od bitu ke qubitu a ještě kousek dál: Kvantové výpočty pro začátečníky", je vynikající. Využívá jemný humor spojený s dvojznačností slova "bit" (v angličtině základní jednotka digitálních dat a v češtině "kousek"), což může být velmi efektivní při zaujetí pozornosti studentů. Tento vtipný jazykový prvek nejen zpřístupňuje téma, ale také vytváří více uvolněnou a přátelskou atmosféru pro prezentaci, což je ideální pro středoškoláky, kteří se s kvantovými výpočty setkávají poprvé.

Something is going on...

- “Quantum computing” is becoming a buzzword.
- Many new startup companies dealing with QC arise, while well-established companies (IBM, Microsoft, Google, ...) create special quantum-oriented divisions and publish quantum roadmaps for years to come.
- Forbes, October 24, 2023: **“Atom Computing Announces Record-Breaking 1,225-Qubit Quantum Computer”**
- Nature News, December 4, 2023: **“IBM releases first-ever 1,000-qubit quantum chip”**
    - “IBM has unveiled the first quantum computer with more than 1,000 qubits. [...] But the company says that it will now shift gears and focus on making its machines more error-resistant rather than larger.”
- What does it all mean? Are quantum computers already running? Can they bring actual advantages? Do they bring them already?

What are we going to discuss today?

Outline

Deterministic states

- **Deterministic system** (or classical system): "something" (a device) that stores **information** by being in one of the finite number of **states**.
- More formally, a deterministic system is as a pair $(\Sigma, \mathbf{X})$, where Σ is a finite nonempty set (**deterministic state set**) whose elements are called **deterministic states** and $\mathbf{X} \in \Sigma$ is the **current state** of the system.
- Classical bit: $\Sigma = \{0, 1\}$
- Electric fan: $\Sigma = \{\text{off}, \text{low}, \text{mid}, \text{high}\}$
- Christmas lights: $\Sigma = \{\text{off}, \text{waves}, \text{sequential}, \text{flash}, \text{twinkle}, \text{glow}, \text{fade}, \text{steady}\}$

Deterministic states as vectors (1)

Bit:

$$|0\rangle = \begin{array}{cl} \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \begin{array}{l} \mathbf{0} \\ 1 \end{array} \end{array} \qquad |1\rangle = \begin{array}{cl} \begin{bmatrix} 0 \\ 1 \end{bmatrix} & \begin{array}{l} 0 \\ \mathbf{1} \end{array} \end{array}$$

Fan:

$$|\mathrm{off}\rangle = \begin{array}{cl} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} & \begin{array}{l} \textbf{off} \\ \text{low} \\ \text{mid} \\ \text{high} \end{array} \end{array} \quad |\mathrm{low}\rangle = \begin{array}{cl} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} & \begin{array}{l} \text{off} \\ \textbf{low} \\ \text{mid} \\ \text{high} \end{array} \end{array} \quad |\mathrm{mid}\rangle = \begin{array}{cl} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} & \begin{array}{l} \text{off} \\ \text{low} \\ \textbf{mid} \\ \text{high} \end{array} \end{array} \quad |\mathrm{high}\rangle = \begin{array}{cl} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} & \begin{array}{l} \text{off} \\ \text{low} \\ \text{mid} \\ \textbf{high} \end{array} \end{array}$$

- $|?\rangle$ is an example of **bra-ket / Dirac notation**.
  - Ubiquitous in quantum mechanics/computing but we won't go into details here.
  - For us today, it's just a label given to the vector.
- Notice the coordinate notation (on the right) reflects the **size** of state set Σ and the **current state**;
- but it quickly becomes cumbersome with the growing size 🤯

Deterministic states as vectors (2)

Bit:

$$|0\rangle = \left[\begin{array}{c}1\\0\end{array}\right]\begin{array}{l}\mathbf{0}\\1\end{array} \qquad |1\rangle = \left[\begin{array}{c}0\\1\end{array}\right]\begin{array}{l}0\\\mathbf{1}\end{array}$$

Fan:

$$|\mathrm{off}\rangle = \left[\begin{array}{c}1\\0\\0\\0\end{array}\right]\begin{array}{l}\textbf{off}\\\text{low}\\\text{mid}\\\text{high}\end{array} \quad |\mathrm{low}\rangle = \left[\begin{array}{c}0\\1\\0\\0\end{array}\right]\begin{array}{l}\text{off}\\\textbf{low}\\\text{mid}\\\text{high}\end{array} \quad |\mathrm{mid}\rangle = \left[\begin{array}{c}0\\0\\1\\0\end{array}\right]\begin{array}{l}\text{off}\\\text{low}\\\textbf{mid}\\\text{high}\end{array} \quad |\mathrm{high}\rangle = \left[\begin{array}{c}0\\0\\0\\1\end{array}\right]\begin{array}{l}\text{off}\\\text{low}\\\text{mid}\\\textbf{high}\end{array}$$

- Deterministic states form a **basis** (specifically the **standard basis**).
  - Important term which you will learn rigorously in your first linear algebra course 😉
  - Basis vectors are **linearly independent**: they *cannot* be expressed as a **linear combination** of each other;
  - while any vector of the given space *is* a linear combination of the basis vectors.
  - Linear combination? 🤔 It's just a weighted sum of vectors, e.g. $0.3\,|0\rangle + 0.7\,|1\rangle$

Probabilistic states

- Suppose we don't know for sure what is the current deterministic state $\mathbf{X}$ of a system.
- Assume, however, that based on some ***a priori* knowledge**, we can assign some probabilities to its N deterministic states $\Sigma = \{S_0, S_1, \ldots, S_{N-1}\}$.
- We can write down the probabilities succinctly using a **probability vector**:

$$|\mathbf{X}\rangle = \begin{array}{c} \begin{bmatrix} x_0 \\ \vdots \\ x_N \end{bmatrix} \end{array} \begin{array}{l} S_0 \\ \vdots \\ S_{N-1} \end{array} = x_0 \,|S_0\rangle + x_1 \,|S_1\rangle + \cdots + x_{N-1} \,|S_{N-1}\rangle \,,$$

$$x_i = P(\mathbf{X} = S_i) \quad \text{for all } i \in \{0, \ldots, N-1\}$$

- All entries of the vector must be **nonnegative real numbers**.
- The **sum** of the (absolute values of the) entries (1-norm or taxicab norm) is equal to one,

$$\|\,|\mathbf{X}\rangle\,\|_1 = \sum_{i=1}^{N} |x_i| = 1$$

Note!

- deterministic (classical) states $\subset$ probabilistic states
- I mean, any deterministic state can be regarded as a probabilistic state!

Measurement

- By **measurement** of a *probabilistic* state we will mean:
  - **Unambiguously recognize** the current deterministic state!
- Measurement updates our knowledge about the system.
- The *probabilistic* state before measurement can be in "superposition" of multiple *deterministic* states;
- upon measurement it "collapses" into one of the *deterministic* states.
- Measurement is often subjective as we will see!
- Measuring again and again doesn't bring anything new.

Example: Probabilistic bit (1)

- Suppose we don't know for sure what is the current deterministic state $\mathbf{X}$ of a bit;
- based on some ***a priori*** **knowledge**, we know just probabilities

$$P(\mathbf{X}=0)=0.75, \qquad P(\mathbf{X}=1)=0.25.$$

- More succinctly – using a **probability vector**:

$$|\mathbf{X}\rangle = \begin{bmatrix}0.75\\0.25\end{bmatrix} = 0.75\begin{bmatrix}1\\0\end{bmatrix} + 0.25\begin{bmatrix}0\\1\end{bmatrix} = 0.75\,|0\rangle + 0.25\,|1\rangle$$

- All entries of the vector are **nonnegative real numbers** and the taxicab norm is 1,

$$\|\,|\mathbf{X}\rangle\,\|_1 = \sum_{i=1}^{n} |x_i| = 0.75 + 0.25 = 1$$

Example: Probabilistic bit (2)

$$|\mathbf{X}\rangle = \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix} = 0.75\,|0\rangle + 0.25\,|1\rangle\,, \quad \|\,|\mathbf{X}\rangle\,\|_1 = 0.75 + 0.25 = 1$$

- **Measuring** the bit $\mathbf{X}$, we update our knowledge and the state collapsed to a deterministic state:

$$|\mathbf{X}\rangle = 0.75\,|0\rangle + 0.25\,|1\rangle \xrightarrow{\text{measure}} \begin{cases} |0\rangle\,, & P = 0.75, \\ |1\rangle\,, & P = 0.25, \end{cases}$$

i.e. $|\mathbf{X}\rangle$ after measurement is

$$|\mathbf{X}\rangle = |0\rangle \text{ with probability } P(\mathbf{X} = 0) = 0.75,$$
$$|\mathbf{X}\rangle = |1\rangle \text{ with probability } P(\mathbf{X} = 1) = 0.25.$$

- The measurement is subjective (e.g. I measured but didn't tell my friend the result).
- Measuring again and again doesn't bring anything new.
  - Of course! E.g. $|0\rangle = 1\,|0\rangle \xrightarrow{\text{measure}} |0\rangle\,,\ P = 1.$

Example: Coin

- Assume a **coin**:

$$\Sigma = \{\text{heads}, \text{tails}\}, \quad |\text{heads}\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad |\text{tails}\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

- A coin just flipped, before looking at the result, is in “superposition”.
- A **fair** coin:

$$|\mathbf{X}\rangle = \begin{array}{c} \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} \end{array} \begin{array}{l} \scriptstyle\text{heads} \\ \scriptstyle\text{tails} \end{array} = 0.5\,|\text{heads}\rangle + 0.5\,|\text{tails}\rangle \xrightarrow{\text{measure}} \begin{cases} |\text{heads}\rangle\,, & P = 0.5, \\ |\text{tails}\rangle\,, & P = 0.5, \end{cases}$$

- An **unfair** coin:

$$|\mathbf{X}\rangle = \begin{array}{c} \begin{bmatrix} 0.49 \\ 0.51 \end{bmatrix} \end{array} \begin{array}{l} \scriptstyle\text{heads} \\ \scriptstyle\text{tails} \end{array} = 0.49\,|\text{heads}\rangle + 0.51\,|\text{tails}\rangle \xrightarrow{\text{measure}} \begin{cases} |\text{heads}\rangle\,, & P = 0.49, \\ |\text{tails}\rangle\,, & P = 0.51, \end{cases}$$

- The measurement is subjective.

Example: Card pile

- You have a **pile of 13 single-colour cards**: $\Sigma = \{2, 3, 4, 5, 6, 7, 8, 9, 10, \mathrm{J}, \mathrm{Q}, \mathrm{K}, \mathrm{A}\}$
- You ask a friend to pick one at random without showing it to you.
- The picked card's state before and after checking which symbol it actually has:

$$|\mathbf{X}\rangle = \frac{1}{13} \begin{array}{c} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \end{array} \begin{array}{l} 2 \\ \vdots \\ \mathrm{A} \end{array} = \frac{1}{13} \left(|2\rangle + \cdots + |\mathrm{A}\rangle \right) \xrightarrow{\text{measure}} \begin{cases} |2\rangle \,, & P = 1/13 \\ \vdots \\ |\mathrm{A}\rangle \,, & P = 1/13 \end{cases}$$

- The measurement is subjective.

Example: Sportka 1

- A single "drum" of a **fair "ball lottery"** (like Šance by Sportka) with 10 numbered balls $\Sigma = \{0, 2, \ldots, 9\}$.
- The state before and after a ball is picked:

$$|\mathbf{X}\rangle = \frac{1}{10} \begin{array}{c} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \end{array} \begin{array}{c} 0 \\ \vdots \\ 9 \end{array} = \frac{1}{10} \left(|0\rangle + \cdots + |9\rangle \right) \xrightarrow{\text{measure}} \begin{cases} |0\rangle \,, & P = 1/10 \\ \vdots & \\ |9\rangle \,, & P = 1/10 \end{cases}$$

- This time, the measurement is **objective**! Before the ball is picked, *nobody* knows the outcome.
- That brings us a little bit closer to quantum systems...

Example: Sportka 2

- A single "drum" of a **brutally unfair "ball lottery"** with 10 numbered balls $\Sigma = \{0, 2, \ldots, 9\}$.
- Spits out only 0 or 9 with probabilities $1/3$ and $2/3$ 😇
- The state before and after a ball is picked:

$$|\mathbf{X}\rangle = \begin{array}{c} \begin{bmatrix} \frac{1}{3} \\ 0 \\ \vdots \\ 0 \\ \frac{2}{3} \end{bmatrix} \end{array} \begin{array}{c} 0 \\ 1 \\ \vdots \\ 8 \\ 9 \end{array} = \frac{1}{3}|0\rangle + \frac{2}{3}|9\rangle \xrightarrow{\text{measure}} \begin{cases} |0\rangle\,, & P = 1/3 \\ |9\rangle\,, & P = 2/3 \end{cases}$$

- Notice the conciseness of the Dirac notation for sparse states.
- Objective measurement.

Example: Mechanical oscillator

TODO

Deterministic operations (1)

- These map a deterministic state to a deterministic state.
- We deal with **linear** operations; such can be expressed as **matrix-vector** product:

$$|f(a)\rangle = M\,|a\rangle$$

- There are just four possible deterministic operations on a bit:

$$M_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad M_4 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix},$$

  corresponding to boolean functions

$$f_1(a) = 0, \quad f_2(a) = a, \quad f_3(a) = \neg a, \quad f_4(a) = 1$$

Deterministic operations (2)

$$|f(a)\rangle = M\,|a\rangle$$

$$M_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \qquad M_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I, \qquad M_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = X, \qquad M_4 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix},$$

$$f_1(a) = 0, \qquad f_2(a) = a, \qquad f_3(a) = \neg a, \qquad f_4(a) = 1$$

- Notice that columns of the matrices are formed by $|0\rangle$ and $|1\rangle$.
- and matrix-vector multiplication with a deterministic state vector just extracts the respective column, e.g.

$$|f_3(1)\rangle = M_3\,|1\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle = |\neg 1\rangle$$

Probabilistic operations (1)

- Probabilistic-to-probabilistic but not deterministic-to-deterministic.
- For example, applying

$$M = \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{bmatrix}$$

  to deterministic state vectors yields

  - $M|0\rangle = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \frac{1}{2}|0\rangle + \frac{1}{2}|1\rangle$
  - $M|1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle$

- Assuming the coin again, using $|\mathrm{heads}\rangle = |0\rangle$ and $|\mathrm{tails}\rangle = |1\rangle$, this means in natural language:

  1. If heads given, flip the coin fairly;
  2. if tails given, turn it to heads.

Probabilistic operations (2)

$$M = \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{bmatrix}$$

So far, so good. It gets a bit less intuitive if a probabilistic state is on input. Having, e.g.,

$$|\mathbf{X}\rangle = \frac{1}{5}|0\rangle + \frac{4}{5}|1\rangle \, ,$$

we get

$$\begin{aligned} M|\mathbf{X}\rangle &= \frac{1}{5}\left(\frac{1}{2}|0\rangle + \frac{1}{2}|1\rangle\right) + \frac{4}{5}|0\rangle \\ &= \frac{9}{10}|0\rangle + \frac{1}{10}|1\rangle \, , \end{aligned}$$

Probabilistic operations (3)

- Matrix M representing operations on probabilistic states must satisfy:
    1. All entries of M are **nonnegative real numbers**.
    2. The **sum** of the entries **in each column** is equal to 1; we can say $||M(:,i)||_1 = 1$ for every column index i.
- This is equivalent to saying **every column** is a **probability vector**.
- Every such matrix is called a **stochastic matrix**.
- A stochastic matrix can be considered a **random choice of deterministic operations**. E.g,

$$M_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad M_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad M_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad M_4 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix},$$

$$M = \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{bmatrix} = \frac{1}{2} M_1 + \frac{1}{2} M_3$$

Operation composition

- Operation composition can be expressed simply as **matrix-matrix multiplication**.
- Matrix multiplication is **associative**: $(M_1 M_2) M_3 = M_1 (M_2 M_3) \quad [= M_1 M_2 M_3]$.
- Hence, applying $M_1, M_2, \ldots, M_n$ in that order can be expressed as a single composed operation
$$M = M_n \cdots M_2 M_1$$
- **Not commutative!**

$$M_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \qquad M_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$M_1 M_3 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \qquad M_3 M_1 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

Outline

Compound classical systems (1)

- Let's have **independent deterministic systems** $\mathbf{X}$ and $\mathbf{Y}$ and their **state sets** Σ and Γ.
- We can take them as a single **compound system** $(\mathbf{X}, \mathbf{Y})$.
- State set of $(\mathbf{X}, \mathbf{Y})$ is then defined as the **Cartesian product**

$$\Sigma \times \Gamma = \{(a, b) : a \in \Sigma, b \in \Gamma\}$$

- More generally, a compound system $(\mathbf{X}_1 \cdots \mathbf{X}_n)$ has a state set $\Sigma_1 \times \cdots \times \Sigma_n$.
- In case of bits, $\Sigma_1 = \cdots = \Sigma_n = \Sigma = \{0, 1\}$ and we often write a state $(a_1, \ldots, a_n) \in \Sigma^n$ as a **bit string** $a_1 \ldots a_n$, e.g. $(0, 1, 0) = 010$.
- For example, for $n = 4$, the compound state set is

$$\Sigma = \{0000, 0001, 0010, \ldots, 1110, 1111\}, \quad |\Sigma| = 2^4 = 16.$$

- Mathematically, it is the same as having a single system with $\Sigma = \{0, 1, \ldots, 15\}$ and writing the labels as binary strings padded with zeros to the length n.

Compound classical systems (2)

- The **Cartesian product** of the state sets transforms into the **Kronecker (tensor) product** of the vectors.
- Vector-vector tensor product works like this in general:

$$a = \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

$$a \otimes b = \begin{bmatrix} a_1 b \\ \vdots \\ a_m b \end{bmatrix} \in \mathbb{R}^{mn}$$

Compound classical systems (3)

- The tensor product works like this for deterministic states (standard basis vectors):

$$|0\rangle = \begin{bmatrix}1\\0\end{bmatrix}\begin{matrix}\mathbf{0}\\1\end{matrix} \qquad |1\rangle = \begin{bmatrix}0\\1\end{bmatrix}\begin{matrix}0\\\mathbf{1}\end{matrix}$$

$$|0\rangle \otimes |0\rangle = \begin{bmatrix}1\\0\end{bmatrix}\begin{matrix}\mathbf{0}\\1\end{matrix} \otimes \begin{bmatrix}1\\0\end{bmatrix}\begin{matrix}\mathbf{0}\\1\end{matrix} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}\begin{matrix}\mathbf{00}\\01\\10\\11\end{matrix} = |00\rangle \qquad |0\rangle \otimes |1\rangle = \begin{bmatrix}1\\0\end{bmatrix}\begin{matrix}\mathbf{0}\\1\end{matrix} \otimes \begin{bmatrix}0\\1\end{bmatrix}\begin{matrix}0\\\mathbf{1}\end{matrix} = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}\begin{matrix}00\\\mathbf{01}\\10\\11\end{matrix} = |01\rangle$$

$$|1\rangle \otimes |0\rangle = \begin{bmatrix}0\\1\end{bmatrix}\begin{matrix}0\\\mathbf{1}\end{matrix} \otimes \begin{bmatrix}1\\0\end{bmatrix}\begin{matrix}\mathbf{0}\\1\end{matrix} = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}\begin{matrix}00\\01\\\mathbf{10}\\11\end{matrix} = |10\rangle \qquad |1\rangle \otimes |1\rangle = \begin{bmatrix}0\\1\end{bmatrix}\begin{matrix}0\\\mathbf{1}\end{matrix} \otimes \begin{bmatrix}0\\1\end{bmatrix}\begin{matrix}0\\\mathbf{1}\end{matrix} = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\begin{matrix}00\\01\\10\\\mathbf{11}\end{matrix} = |11\rangle$$

- $|ab\rangle$, $|a\rangle\,|b\rangle$, $|a\rangle \otimes |b\rangle$ hence mean all the same.
- Tensor product means **independence**; the opposite case of dependence comes in a while ....

Compound probabilistic system example: Lottery revisited

- Five "drums" of a fair Šance by Sportka, each with 10 numbered balls $\Sigma = \{0, 2, \ldots, 9\}$.
- The state before and after a ball is picked:

$$
|\mathbf{X}\rangle = \frac{1}{10^5} \begin{array}{c} \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \end{array} \begin{array}{l} 00000 \\ 00001 \\ 00002 \\ \vdots \\ 99999 \end{array} = \frac{1}{10^5} \left(|00000\rangle + \cdots + |99999\rangle \right) \xrightarrow{\text{measure}} \begin{cases} |00000\rangle\,, & P = 1/10^5 \\ \vdots & \\ |99999\rangle\,, & P = 1/10^5 \end{cases}
$$

- An unfair "sparse" Šance spitting one and only one 1 and the rest are 0s:

$$
|\mathbf{X}\rangle = \frac{1}{5} \left(|00001\rangle + |00010\rangle + |00100\rangle + |01000\rangle + |10000\rangle \right) \xrightarrow{\text{measure}} \begin{cases} |00001\rangle\,, & P = 1/5 \\ |00010\rangle\,, & P = 1/5 \\ |00100\rangle\,, & P = 1/5 \\ |01000\rangle\,, & P = 1/5 \\ |10000\rangle\,, & P = 1/5 \end{cases}
$$

Independent vs correlated systems (1)

- Individual states of a compound deterministic state, like $|0\rangle$ and $|1\rangle$ in $|01\rangle$, are independent "by construction".
- Independence is less obvious in the case of probabilistic states such as compound system $(\mathbf{X}, \mathbf{Y})$ with state vector

$$|\mathbf{XY}\rangle = \frac{1}{6}|00\rangle + \frac{1}{12}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{4}|11\rangle$$

- $\mathbf{X}$ and $\mathbf{Y}$ with state sets Σ and Γ are independent if and only if

$$\forall a \in \Sigma, b \in \Gamma : \quad P((\mathbf{X}, \mathbf{Y}) = (a, b)) = P(\mathbf{X} = a)P(\mathbf{Y} = b).$$

Independent vs correlated systems (2)

$$
\begin{aligned}
|\mathbf{XY}\rangle &= \frac{1}{6}|00\rangle + \frac{1}{12}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{4}|11\rangle \\
P(\mathbf{XY}=01) &= \frac{1}{12}, \\
P(\mathbf{X}=0) &= P(\mathbf{XY}=00) + P(\mathbf{XY}=01) = \frac{1}{6} + \frac{1}{12} = \frac{1}{4}, \\
P(\mathbf{Y}=1) &= P(\mathbf{XY}=01) + P(\mathbf{XY}=11) = \frac{1}{12} + \frac{1}{4} = \frac{1}{3},
\end{aligned}
$$

so indeed $P(\mathbf{XY}=01) = P(\mathbf{X}=0)P(\mathbf{Y}=1)$ and the same can be shown for the other combinations $00, 10, 11$. More succinctly, we can just argue that

$$|\mathbf{XY}\rangle = |\mathbf{X}\rangle \otimes |\mathbf{Y}\rangle, \text{ where}$$

$$|\mathbf{X}\rangle = \frac{1}{4}|0\rangle + \frac{3}{4}|1\rangle \quad \text{and} \quad |\mathbf{Y}\rangle = \frac{2}{3}|0\rangle + \frac{1}{3}|1\rangle.$$

Independent vs correlated systems (3)

Think of two coins glued together...

$$
\begin{aligned}
|\mathbf{XY}\rangle &= \frac{1}{2}\,|00\rangle + \frac{1}{2}\,|11\rangle \\
P(\mathbf{XY} = 01) &= 0, \\
P(\mathbf{X} = 0) &= P(\mathbf{XY} = 00) = \frac{1}{2}, \\
P(\mathbf{Y} = 1) &= P(\mathbf{XY} = 11) = \frac{1}{2},
\end{aligned}
$$

This system can't be independent because

$$
P(\mathbf{XY} = 01) = 0 \quad \neq \quad \frac{1}{4} = P(\mathbf{X} = 0)P(\mathbf{Y} = 1)
$$

The **lack of independence** means that $\mathbf{X}$ and $\mathbf{Y}$ are **correlated**.
Alternatively, we can argue that there are no $|\mathbf{X}\rangle$, $|\mathbf{Y}\rangle$ such that $|\mathbf{XY}\rangle = |\mathbf{X}\rangle \otimes |\mathbf{Y}\rangle$.

Full measurement

If we measure all subsystems of a compound system at once, there's actually no difference to the single system state. For example:

$$\frac{1}{6}|00\rangle + \frac{1}{12}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{4}|11\rangle \xrightarrow{\text{measure}} \begin{cases} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{cases} \quad P = \begin{cases} 1/6 \\ 1/12 \\ 1/2 \\ 1/4 \end{cases}$$

$$\frac{1}{2}|0\rangle + \frac{1}{2}|1\rangle \xrightarrow{\text{measure}} \begin{cases} |0\rangle \\ |1\rangle \end{cases} \quad P = \begin{cases} 1/2 \\ 1/2 \end{cases}$$

$$\frac{1}{2}|00\rangle + \frac{1}{2}|11\rangle \xrightarrow{\text{measure}} \begin{cases} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{cases} \quad P = \begin{cases} 1/2 \\ 0 \\ 0 \\ 1/2 \end{cases}$$

Partial measurement (1)

We apply the usual conditional and marginal probability formulas

$$P(\mathbf{Y} = b \,|\, \mathbf{X} = a) = \frac{P(\mathbf{XY} = ab)}{P(\mathbf{X} = a)}, \qquad P(\mathbf{X} = a) = \sum_b P((\mathbf{XY} = ab).$$

First bit:

$$\begin{aligned}
&\frac{1}{6}\,|00\rangle + \frac{1}{12}\,|01\rangle + \frac{1}{2}\,|10\rangle + \frac{1}{4}\,|11\rangle \\
&= |0\rangle \otimes \left(\frac{1}{6}\,|0\rangle + \frac{1}{12}\,|1\rangle\right) \;+\; |1\rangle \otimes \left(\frac{1}{2}\,|0\rangle + \frac{1}{4}\,|1\rangle\right) \\
&= |0\rangle \otimes |r_{0,0}\rangle + |1\rangle \otimes |r_{0,1}\rangle \\
&\xrightarrow{\text{measure}} \begin{cases} |0\rangle \otimes \frac{|r_{0,0}\rangle}{\||r_{0,0}\rangle\|_1} = |0\rangle \otimes \left(\frac{2}{3}\,|0\rangle + \frac{1}{3}\,|1\rangle\right), & P = \|\,|r_{0,0}\rangle\,\|_1 = 1/4 \\ |1\rangle \otimes \frac{|r_{0,1}\rangle}{\||r_{0,1}\rangle\|_1} = |1\rangle \otimes \left(\frac{2}{3}\,|0\rangle + \frac{1}{3}\,|1\rangle\right), & P = \|\,|r_{0,1}\rangle\,\|_1 = 3/4 \end{cases}
\end{aligned}$$

Partial measurement (2)

Second bit:

$$
\begin{aligned}
&\frac{1}{6}\left|00\right\rangle + \frac{1}{12}\left|01\right\rangle + \frac{1}{2}\left|10\right\rangle + \frac{1}{4}\left|11\right\rangle \\
&= \left(\frac{1}{6}\left|0\right\rangle + \frac{1}{2}\left|1\right\rangle\right) \otimes \left|0\right\rangle \; + \; \left(\frac{1}{12}\left|0\right\rangle + \frac{1}{4}\left|1\right\rangle\right) \otimes \left|1\right\rangle \\
&= \left|r_{1,0}\right\rangle \otimes \left|0\right\rangle + \left|r_{1,1}\right\rangle \otimes \left|1\right\rangle \\
&\xrightarrow{\text{measure}} \begin{cases} \frac{\left|r_{1,0}\right\rangle}{\|\left|r_{1,0}\right\rangle\|_1} \otimes \left|0\right\rangle = \left(\frac{1}{4}\left|0\right\rangle + \frac{3}{4}\left|1\right\rangle\right) \otimes \left|0\right\rangle, \quad P = \|\left|r_{1,0}\right\rangle\|_1 = 2/3 \\ \frac{\left|r_{1,1}\right\rangle}{\|\left|r_{1,1}\right\rangle\|_1} \otimes \left|1\right\rangle = \left(\frac{1}{4}\left|0\right\rangle + \frac{3}{4}\left|1\right\rangle\right) \otimes \left|1\right\rangle, \quad P = \|\left|r_{1,1}\right\rangle\|_1 = 1/3 \end{cases}
\end{aligned}
$$

Operations on multiple systems (1)

- Corresponding to independent or correlated probabilistic states, we can have independent or collective operations on compound states.
- Independence is again expressed with the tensor product $\otimes$.

Example – negate the first bit and do nothing to the other:

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad X \otimes I = \begin{bmatrix} O_2 & I_2 \\ I_2 & O_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$(X \otimes I)\,|10\rangle = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = |00\rangle\,, \quad \text{or, using distributivity,}$$

$$(X \otimes I)\,|10\rangle = (X\,|1\rangle) \otimes (I\,|0\rangle) = |00\rangle$$

Operations on multiple systems (2)

- We can also have operations that act collectively on multiple subsystems (bits) and, hence, can't be decomposed using $\otimes$.

Example – controlled NOT for 2-bit system $\mathbf{XY}$; if $\mathbf{X}$ is 1, negate $\mathbf{Y}$, else no-op:

$$CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$CNOT\,|00\rangle = |00\rangle \qquad CNOT\,|01\rangle = |01\rangle$$
$$CNOT\,|10\rangle = |11\rangle \qquad CNOT\,|11\rangle = |10\rangle$$

Outline

From probabilistic to quantum (1)

- We can generalize **probabilistic states to quantum states** quite naturally!
- Assume $|\psi\rangle$ is a quantum state, $|\mathbf{X}\rangle$ is the corresponding probabilistic state.
  - Indefinite number of quantum states map to the same probabilistic state.
- Recall $|\mathbf{X}\rangle$ consists of **nonnegative real** coefficients, and its **taxicab norm is 1**.
- **Quantum states** emerge by attaching **phases**:

$$|\mathbf{X}\rangle = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in (\mathbb{R}_0^+)^n, \ \| \, |\mathbf{X}\rangle \, \|_1 = 1, \qquad |\varphi\rangle = \begin{bmatrix} \varphi_1 \\ \vdots \\ \varphi_n \end{bmatrix} \in [0, 2\pi)^n, \qquad \psi = (|\mathbf{X}\rangle \, , |\varphi\rangle)$$

- **That's essentially it!** This is the main difference and main source of all the hopes about quantum computing!
- The underlying deterministic state set can be the same – no difference.

From probabilistic to quantum (2)

$$|\mathbf{X}\rangle = [x_1, \ldots, x_n]^T \in (\mathbb{R}_0^+)^n, \quad \| \, |\mathbf{X}\rangle \, \|_1 = 1, \quad |\varphi\rangle = [\varphi_1, \ldots, \varphi_n]^T \in [0, 2\pi)^n, \quad \psi = (|\mathbf{X}\rangle, |\varphi\rangle)$$

- It's, however, more common and handy to express a quantum state as a **complex vector**!
- Each entry ψ_i is a complex **probability amplitude**, encoding both **magnitude** $|\psi_i| = \sqrt{x_i}$ and **phase** $\varphi_i = \arg(\psi_i)$, i.e.,

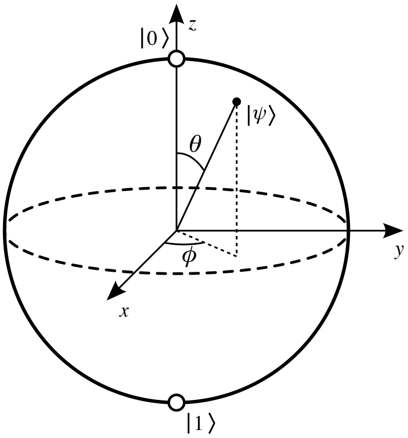
$$\begin{aligned} |\psi\rangle &= [\psi_1, \ldots, \psi_n]^T \in \mathbb{C}^n, \\ \psi_i &= |\psi_i|(\cos\varphi_i + i \sin\varphi_i) \quad \text{for } i = 1, \ldots, n. \end{aligned}$$

- Measurement probabilities are then given as $P_i = x_i = |\psi_i|^{\mathbf{2}}$ (details in a bit).
- The normalization condition uses a different norm – the **Euclidean norm**:

$$\| \, |\psi\rangle \, \|_{\mathbf{2}} = \left(\sum_{i=1}^{n} |\psi_i|^{\mathbf{2}} \right)^{1/\mathbf{2}} = 1$$

- **Composition** of independent states works the same: using **tensor product** $\otimes$ again!

Bloch sphere



Quantum measurement

- **Measuring** probabilistic and quantum states is **similar but not same** from mathematical viewpoint.
  - Just need to keep in mind we now use **2-norm** and **complex** coefficients;
  - coefficients are **amplitudes**, not probabilities directly;
  - probabilities are given by **absolute values squared** of amplitudes;
  - but in the end, we obtain **classical deterministic information** again!
- What is very **different in physical reality**, though, is that quantum measurements are always **objective** and change the state **for everyone/everything**!
- We say the state gets **collapsed**.
  - Leaking information leads to collapse.
  - It doesn't matter who or what receives the information 👻

Quantum measurement

V **You**

draw collapse of quantum state

ChatGPT

Here is an abstract representation of the collapse of a quantum state, blending scientific and artistic elements to depict the transition from quantum superposition to a definite state. The image captures the mystical and enigmatic nature of quantum mechanics.

Full quantum measurement (1)

$$\left.\begin{array}{l} \frac{1}{\sqrt{2}}\,|0\rangle \;-\; \frac{1}{\sqrt{2}}\,|1\rangle \\ \frac{1}{\sqrt{2}}\,|0\rangle \;+\; \frac{1}{\sqrt{2}}\,|1\rangle \\ \frac{i}{\sqrt{2}}\,|0\rangle \;-\; \frac{1+i}{2}\,|1\rangle \end{array}\right\} \xrightarrow{\text{measure}} \begin{cases} |0\rangle\,, & P = 1/2 \\ |1\rangle\,, & P = 1/2 \end{cases}$$

- These are considered **different states** but give the **same probabilities**!
- In other words, they map to the same probabilistic state.
- They are **not distinguishable** by standard basis measurement.

Full quantum measurement (2)

$$\left.\begin{array}{l}\frac{1}{\sqrt{6}}\left|00\right\rangle + \frac{1}{\sqrt{12}}\left|01\right\rangle + \frac{1}{\sqrt{2}}\left|10\right\rangle + \frac{1}{2}\left|11\right\rangle \\ \frac{i}{\sqrt{6}}\left|00\right\rangle - \frac{e^{0.1234i}}{\sqrt{12}}\left|01\right\rangle + \frac{\sqrt{3}-i}{2\sqrt{2}}\left|10\right\rangle + \frac{1+\sqrt{3}i}{4}\left|11\right\rangle\end{array}\right\} \xrightarrow{\text{measure}} \begin{cases}\left|00\right\rangle, & P = 1/6 \\ \left|01\right\rangle, & P = 1/12 \\ \left|10\right\rangle, & P = 1/2 \\ \left|11\right\rangle, & P = 1/4\end{cases}$$

- These are considered **different states** but give the **same probabilities**!
- In other words, they map to the same probabilistic state.
- They are **not distinguishable** by standard basis measurement.

Partial quantum measurement

Like for probabilistic, just with $\|.\|_2$.
Second qubit:

$$
\begin{aligned}
|\phi\rangle &= \frac{1}{\sqrt{6}}|00\rangle + \frac{1}{\sqrt{12}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle + \frac{1}{2}|11\rangle \\
&= \left(\frac{1}{\sqrt{6}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) \otimes |0\rangle \ + \ \left(\frac{1}{\sqrt{12}}|0\rangle + \frac{1}{2}|1\rangle\right) \otimes |1\rangle \\
&= |r_{1,0}\rangle \otimes |0\rangle + |r_{1,1}\rangle \otimes |1\rangle \\
&\xrightarrow{\text{measure}} \begin{cases} \dfrac{|r_{1,0}\rangle}{\|\,|r_{1,0}\rangle\,\|_2} \otimes |0\rangle = \left(\dfrac{1}{2}|0\rangle + \dfrac{\sqrt{3}}{2}|1\rangle\right) \otimes |0\rangle\,, & P = \|\,|r_{1,0}\rangle\,\|_2^2 = 2/3 \\ \dfrac{|r_{1,1}\rangle}{\|\,|r_{1,1}\rangle\,\|_2} \otimes |1\rangle = \left(\dfrac{1}{2}|0\rangle + \dfrac{\sqrt{3}}{2}|1\rangle\right) \otimes |1\rangle\,, & P = \|\,|r_{1,1}\rangle\,\|_2^2 = 1/3 \end{cases}
\end{aligned}
$$

Vsuvka: komplexní sdružení a skalární součin, ortogonální a unitární matice

- **komplexně sdružené číslo** k číslu $z = a + bi = |z|e^{i\phi}$ se nazývá číslo $\overline{z} = a - bi = |z|e^{-i\phi}$
  - Vznikne tedy překlopením znaménka u imaginární části.
  - obrázek a příklad
- **skalární součin v komplexním oboru**:
  - $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = \mathbf{u}^* \mathbf{v} = \overline{u_1} v_1 + \cdots + \overline{u_n} v_n$
  - * značí hermitovskou (komplexně sdruženou) transpozici: $\mathbf{A}^* = \overline{\mathbf{A}^T}$ (místo * se taky používá †, ^{H}, $^+$)
  - Je-li $\mathbf{u} = |\psi\rangle$ a $\mathbf{v} = |\phi\rangle$, značíme $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \psi | \phi \rangle$
  - Mimochodem "bra" vektor se definuje $\langle \psi | = |\psi\rangle^*$
- **ortogonální matice** je čtvercová matice $\mathbf{A}$: $\mathbf{A}^T \mathbf{A} = \mathbf{I} = \mathbf{A}\mathbf{A}^T$
- **unitární matice** je čtvercová matice $\mathbf{A}$: $\mathbf{A}^* \mathbf{A} = \mathbf{I} = \mathbf{A}\mathbf{A}^*$

Quantum operations

- Quantum operations need to be **unitary** rather than **stochastic**:

$$MM^\dagger = M^\dagger M = I$$

- Equivalent to the requirement (again) that each column must be a valid state ($\|.\|_2 = 1$).
- This time also each row.
- Usually called **quantum gates**.

Some important operations

Pauli matrices:

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

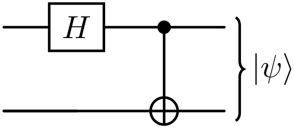
Hadamard:

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Controlled NOT, SWAP:

$$\mathsf{CNOT}_{0,1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad \mathsf{CNOT}_{1,0} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \mathsf{SWAP} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

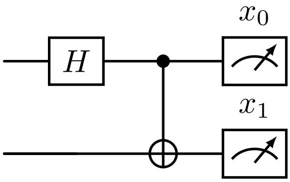
Quantum circuit



$$
\begin{aligned}
|\psi\rangle &= \mathsf{CNOT}_{0,1}(I \otimes H)\,|00\rangle \\
&= \mathsf{CNOT}_{0,1}\left(|0\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right) \\
&= \mathsf{CNOT}_{0,1}\left(\frac{1}{\sqrt{2}}(|00\rangle + |01\rangle)\right) \\
&= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = |\phi^+\rangle
\end{aligned}
$$

- Graphical representation of unitary gates.
- Implicit initialization to $|0\rangle$.
- Gates drawn in the order of application, i.e., reversely to mathematical notation.
- Horizontal lines $=$ time (more to right $=$ later).
- Parallel lines $=$ tensor product; lines joined $=$ collective operation.
- Qiskit convention: topmost qubit in circuit $=$ rightmost in ket $= q_0$

Quantum circuit with measurement



$$|\phi^+\rangle \xrightarrow{\text{measure}} |x_1 x_0\rangle = \begin{cases} |00\rangle\,, & P = 1/2 \\ |11\rangle\,, & P = 1/2 \end{cases}$$

Bell states and entanglement

Probabilistic state:

$$\frac{1}{2}|00\rangle + \frac{1}{2}|11\rangle$$

Bell quantum states:

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle \qquad |\psi^+\rangle = \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle$$

$$|\phi^-\rangle = \frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|11\rangle \qquad |\psi^+\rangle = \frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle$$

- Bell states are schoolbook examples of entangled states.
- Correlation of probabilistic states maps to **entanglement** in the quantum world.
- In our simplistic formulation, **entanglement = correlation**.
- However, in physical reality, entanglement is a much **more powerful** concept.
- Leads to phenomena without classical counterparts, such as **quantum teleportation**.

Outline

Why bother? (1)

- Qubits are much more powerful than old good bits, especially when they "cooperate".
    - Information can be stored not only in the basis vectors (downgrade to deterministic!) but also in the amplitudes (complex numbers)!
    - A qubit is a continuum; the amplitudes have basically infinite precision.

$$\alpha_0 |000\rangle + \alpha_1 |001\rangle + \alpha_2 |010\rangle + \cdots + \alpha_7 |111\rangle$$

  That's 8 complex numbers vs. integers $0, \ldots, 7$ of classical information!

- If we're able to map our DOFs to the amplitudes $\Rightarrow$ exponential "storage"!
- There are also algorithms already known which bring exponential speedup in the number of operations.

Question: How many qubits do you need to represent every mm^3 of the whole Earth as a quantum amplitude?

Answer

https://nssdc.gsfc.nasa.gov/planetary/factsheet/earthfact.html

108.321	1.08321E+12	1.08321E+30	99.77315581
10^{10} km^3	km^3	mm^3	log2

Why bother? (2)

Quantum simulations:

- Simulating elementary particles is exponentially expensive.
- It's hard to simulate even tens of atoms on classical (super)computers.
- Hundreds impossible even for all today's computers working together!
    - There are "only" 10^{82} atoms in the known universe!
- Quantum computers scale linearly because they "are" the elementary particles.
- Big potential also outside of quantum physics / chemistry.

Why bother? (3)

- Intrinsic guaranteed randomness.
    - It's hard to implement something like a fair coin on the computer bit level...
    - All random number generators on classical computers are actually pseudo-random!
    - QC allows us to prepare distributions from which we sample by measuring.
    - Intuitively suitable for any probabilistic approach...
- Cryptography.
    - Guaranteed randomness!
    - On the one hand, QC brings exponentially faster algorithms for integer prime factorization! Potential to crack current cyphers.
    - On the other hand, new opportunities for unbreakable safe communication.
- New communication protocols or even means of communication via quantum teleportation.

Limitations

- There are important limitations, though!
- Readout problem!
  - The amplitudes just represent a distribution from which we sample.
  - Reading a qubit collapses it, and we must start over.
  - Estimating the full state is exponentially expensive.
- Current machines are noisy!
  - This mainly means we need redundancy in qubit count / circuit depth, so it holds us back.

Kernbotschaften zum Mitnehmen 🥡 (1)

A quantum computer is a weird Sportka, where

- you can have currently ~ 1000 balls,
- each ball behaves like a Bloch sphere with two possible outcomes,
- you can manipulate the complex magnitudes and phases of the individual spheres,
- you can correlate the spheres as you wish.

Even though this is hardly implementable in reality, it is still an extremely simplistic and less powerful beast than a real quantum machine!

- E.g., entanglement works at any distance.

Kernbotschaften zum Mitnehmen 🥡 (2)

The ingredients we need are mainly

- complex numbers,
- (complex) linear algebra,
- simple probability theory.

This is not the only model of quantum computation!

- I just described (quite superficially) the **quantum/unitary gate/operator** model.
- It's a **model describing quantum information** and the **basic programming model for quantum computers** dictated by the fundamental rules of quantum mechanics.
- It's **not** the **most general** model, but sufficient in many cases.
- A more general (and more involved) description of quantum information is the **density matrix model**, which we don't cover today.

Credits

- Inspired by the IBM Quantum Learning course **Basics of Quantum Information** by Prof. John Watrous, Technical Director, IBM Quantum Education
- Some good thoughts also in **Quantum Country**
- Bible of QC = Nielsen and Chuang: **Quantum Computation and Quantum Information**

First quantum algorithm: Deutsch's algorithm (1)

- Assume function $f : \{0,1\} \to \{0,1\}$. There are only 4 of them:

k	$f_k(0)$	$f_k(1)$	name	f_k type
0	0	0	zero	0
1	0	1	id	1
2	1	0	neg	1
3	1	1	one	0

(0 = constant, 1 = balanced)

- Deutsch's problem:
  Input: function $f : \{0,1\} \to \{0,1\}$
  Output: type of f

First quantum algorithm: Deutsch's algorithm (2)

- In quantum world, we work with unitaries.
- We can map any boolean function f to a unitary U_f which works like this:

$$U_f \left|y\right\rangle \left|x\right\rangle = \left|y \oplus f(x)\right\rangle \left|x\right\rangle$$

$\|yx\rangle$	$f_0(x)$	$\|y \oplus f_0(x)\rangle \|x\rangle$
$\|00\rangle$	0	$\|00\rangle$
$\|01\rangle$	0	$\|01\rangle$
$\|10\rangle$	0	$\|10\rangle$
$\|11\rangle$	0	$\|11\rangle$

$$U_{f_0} = \left[\left|00\right\rangle, \left|01\right\rangle, \left|10\right\rangle, \left|11\right\rangle\right] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I_4$$

$\|yx\rangle$	$f_1(x)$	$\|y \oplus f_1(x)\rangle \|x\rangle$
$\|00\rangle$	0	$\|00\rangle$
$\|01\rangle$	1	$\|11\rangle$
$\|10\rangle$	0	$\|10\rangle$
$\|11\rangle$	1	$\|01\rangle$

$$U_{f_1} = \left[\left|00\right\rangle, \left|11\right\rangle, \left|10\right\rangle, \left|01\right\rangle\right] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \mathsf{CNOT}_{0,1}$$

First quantum algorithm: Deutsch's algorithm (3)

- In quantum world, we work with unitaries.
- We can map any boolean function f to a unitary U_f which works like this:

$$U_f |y\rangle |x\rangle = |y \oplus f(x)\rangle |x\rangle$$

$\lvert yx\rangle$	$f_2(x)$	$\lvert y \oplus f_2(x)\rangle \lvert x\rangle$
$\lvert 00\rangle$	1	$\lvert 10\rangle$
$\lvert 01\rangle$	0	$\lvert 01\rangle$
$\lvert 10\rangle$	1	$\lvert 00\rangle$
$\lvert 11\rangle$	0	$\lvert 11\rangle$

$$U_{f_2} = \left[|10\rangle, |01\rangle, |00\rangle, |11\rangle\right] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

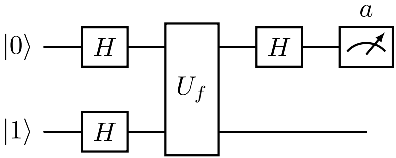
$$= (X \otimes I_2)\mathsf{CNOT}_{0,1}$$

$\lvert yx\rangle$	$f_3(x)$	$\lvert y \oplus f_3(x)\rangle \lvert x\rangle$
$\lvert 00\rangle$	1	$\lvert 10\rangle$
$\lvert 01\rangle$	1	$\lvert 11\rangle$
$\lvert 10\rangle$	1	$\lvert 00\rangle$
$\lvert 11\rangle$	1	$\lvert 01\rangle$

$$U_{f_3} = \left[|10\rangle, |11\rangle, |00\rangle, |01\rangle\right] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = X \otimes I_2$$

First quantum algorithm: Deutsch's algorithm (4)

- Original Deutsch's problem:
  Input: function $f : \{0,1\} \to \{0,1\}$
  Output: type of f
- Equivalent problem:
  Input: unitary $U_f : \mathbb{R}^{4\times 4} \to \mathbb{R}^{4\times 4}$, $U_f \left|y\right\rangle \left|x\right\rangle = \left|y \oplus f(x)\right\rangle \left|x\right\rangle$
  Output: type of f that U_f represents
- Deutsch's algorithm in circuit form:



- Matrix form: $(I \otimes H)U_f(H \otimes H)\left|10\right\rangle \xrightarrow{\text{measure}} \left|?a\right\rangle,\ a = 0, 1$
- This algorithm, specifically the result of measurement of the upper qubit, will yield $a = 0$ if f is constant and $a = 1$ if f is balanced.

TODOs

- Irrelevance of global phase
- How to build a QC and the trade-off between stable quantum properties and ability to interact with it (control and measure)
- Decoherence = quantum noise = unwanted interactions with the outside world