The inverse problem for the permittivity

Julia I. M. Hauser

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Assumptions

$$-\nabla \cdot (\sigma \nabla u) = 0 \text{ in } \Omega$$

Assumptions:

- Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded domain with Lipschitz boundary $\Gamma := \partial\Omega$
- $\sigma \in C^2(\overline{\Omega})$
- $\exists \sigma_0 \in \mathbb{R}_+ : \sigma_0^{-1} \geq \sigma(x) \geq \sigma_0 > 0$
- Let $\gamma_0^{int}\sigma = \sigma_\Gamma$ be known

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The original problem

Find $u \in H^1(\Omega)$ such that

$$\begin{aligned} -\nabla \cdot (\sigma \nabla u) &= 0 && \text{in } \Omega, \\ u &= g && \text{on } \Gamma, \end{aligned}$$

where $g \in H^{1/2}(\Gamma)$. This problem is uniquely solvable.

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The second problem

With the transform $v_\sigma = \sigma^{1/2} u$ one can rewrite the equation

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into

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with

$$q = \frac{\Delta \sigma^{1/2}}{\sigma^{1/2}}.$$

Poincaré-Steklov operator S

We define the Poincaré-Steklov operators (DtN operators)

$$S_\sigma g := \sigma \frac{\partial u}{\partial \nu}\Big|_\Gamma \quad \text{and} \quad \tilde{S}_q h := \frac{\partial v_\sigma}{\partial \nu}\Big|_\Gamma,$$

where u solves

$$\begin{aligned} -\nabla \cdot (\sigma \nabla u) &= 0 && \text{in } \Omega, \\ u &= g && \text{on } \Gamma, \end{aligned}$$

and v_σ solves

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The problem

We want to compute σ with the Poincaré-Steklov operators S for given boundary data g.

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Uniqueness

- For $d = 2$ the uniqueness was proven in [K.Astala L.Päivärinta. 2006].
- Let $d \geq 3$.

Theorem

Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and $\sigma_1, \sigma_2 \in C^2(\overline{\Omega})$ be two positive functions. If $S_{\sigma_1} = S_{\sigma_2}$, then $\sigma_1 = \sigma_2$ almost everywhere in Ω.

[M. Salo 2008, Calderon problem, Lecture notes]

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Uniqueness

Theorem

Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and $q_1, q_2 \in L^\infty(\Omega)$, for which the Dirichlet problems $(-\Delta + q_1)u_1 = 0$ and $(-\Delta + q_2)u_2 = 0$ in Ω are uniquely solvable in a weak sense. If $\tilde{S}_{q_1} = \tilde{S}_{q_2}$, then $q_1 = q_2$ in Ω almost everywhere.

[M. Salo 2008, Calderon problem, Lecture notes]

Goal

We want to derive an algorithm to compute σ with the Poincaré-Steklov Operators $\tilde{S}$ for given boundary data g.

We want to test this algorithm with a reconstruction problem.

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The algorithm[1]

Now we go through the following steps

$$g \to \tilde{S}_q \to q \to \sigma$$

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The algorithm

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Extension of the Schrödinger equation

Let $\eta \in \mathbb{C}^d$ with $\eta \cdot \eta = 0$. Consider the extension of $v_\sigma = \sigma^{1/2} u$ which solves

$$\begin{aligned} (-\Delta + \tilde{q})\psi_\eta &= 0 && \text{in } \mathbb{R}^d \setminus \Gamma, \\ \psi_\eta &= \sigma^{1/2} g && \text{on } \Gamma, \\ \psi_\eta &= e^{i\eta \cdot x} + O(\frac{1}{|x|}) && \text{for } x \to \infty, \end{aligned}$$

where

$$\tilde{q}(x) = \begin{cases} \frac{\Delta \sigma^{1/2}}{\sigma^{1/2}} & \text{in } \Omega, \\ 0 & \text{otherwise.} \end{cases}$$

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\tilde{q}(x) = \begin{cases} \frac{\Delta \sigma^{1/2}}{\sigma^{1/2}} & \text{in } \Omega, \\ 0 & \text{otherwise.} \end{cases}
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With the previous equation we get the following equation for the DtN Operator $\tilde{S}_q$

$$V\tilde{S}_q\psi_\eta(x) = e^{i\eta\cdot x} - (\frac{1}{2}I - K)\sigma^{1/2}g(x).$$

for a.e. $x \in \Gamma$.

Computation of $\tilde{S}_q\psi_\eta$

The operators V and K are defined as follows:

$$V : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma),$$
$$K : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma)$$

with

$$Vv(x) := -\frac{1}{2\pi}\int_\Gamma \log(|x-y|)v(y)\, ds_y,$$
$$Kw(x) := -\frac{1}{2\pi}\int_\Gamma \frac{\partial}{\partial n_y}\log(|x-y|)w(y)\, ds_y.$$

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The algorithm

$$g \to \tilde{S}_q \to q \to \sigma$$

Computation of q

Let

$$V_\xi := \{\eta \in \mathbb{C}^d | \eta \cdot \eta = (\xi + \eta) \cdot (\xi + \eta) = 0\}.$$

It holds that

$$\mathcal{F}\tilde{q}(\xi) = \lim_{\substack{|\eta| \to \infty \\ \eta \in V_\xi}} \int_\Gamma e_{-(\xi+\eta)} \left(\tilde{S}_q \psi_\eta - \tilde{S}_0 v_1 \right) \, ds_x$$

where $e_\eta(x) := e^{i\eta \cdot x}$ and v_1 solves the equation

$$\begin{aligned} -\Delta v_1 &= 0 && \text{in } \Omega, \\ v_1 &= \sigma^{1/2} g && \text{on } \Gamma. \end{aligned}$$

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Computation q

Then we compute

$$q(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} (\mathcal{F}q)(\xi) e^{i\xi\cdot x}\, d\xi.$$

The algorithm

$$g \to \tilde{S}_q \to q \to \sigma$$

Computation of σ

The permittivity σ solves

$$\begin{aligned} (-\Delta + q)\sigma^{1/2} &= 0 && \text{in } \Omega, \\ \sigma^{1/2} &= \sigma_\Gamma^{1/2} && \text{on } \Gamma, \end{aligned}$$

because $q = \frac{\Delta\sigma^{1/2}}{\sigma^{1/2}}$.

The discrete reconstruction problem

The algorithm

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Computation of g and $\tilde{S}_q\psi_\eta$- discretization

Let $u \in S_h^1(\Omega)$ and $\tilde{S}_q\psi_\eta =: t \in S_h^0(\Gamma)$. Furthermore let $\{\phi_j^1\}_{j=1}^N$ be the basis of $S_h^1(\Omega)$ and $\{\phi_k^0\}_{k=1}^M$ the basis of $S_h^0(\Gamma)$. We consider the equation

$$\begin{pmatrix} A_{II} & A_{\Gamma I} & \\ A_{I\Gamma} & A_{\Gamma\Gamma} & -M_h^\top \\ & (\frac{1}{2}M_h - K_h)\Sigma & V_h \end{pmatrix} \begin{pmatrix} u_I \\ u_\Gamma \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ f_\eta \end{pmatrix},$$

where

$$\begin{aligned} u[j] &:= \langle u, \phi_j^1 \rangle_{L^2(\Omega)} & \forall j &= 1, ..., N, \\ t[l] &:= \langle \tilde{S}_q\psi_\eta, \phi_l^0 \rangle_\Gamma & \forall l &= 1, ..., M, \\ f_\eta[k] &:= \langle e_\eta, \phi_k^0 \rangle_\Gamma & \forall k &= 1, ..., M, \end{aligned}$$

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$$\begin{aligned} A[i,j] &:= \langle \sigma \nabla \phi_j^1, \nabla \phi_i^1 \rangle_{L^2(\Omega)} && \forall i,j = 1,...,N, \\ V_h[k,l] &:= \langle V\phi_l^0, \phi_k^0 \rangle_\Gamma && \forall k,l = 1,...M, \\ K_h[k,l] &:= \langle K\phi_l^0, \phi_k^0 \rangle_\Gamma && \forall k,l = 1,...M, \\ M_h[k,l] &:= \langle \phi_l^0, \phi_k^0 \rangle_\Gamma && \forall k,l = 1,...M, \\ \Sigma[k,k] &:= \sigma^{1/2}(x_k^b) && \forall k = 1,...M, \end{aligned}$$

The algorithm

$$\sigma \to g \to \tilde{S}_q \to q \to \sigma$$

Computation q - discretization

Choose $\tilde{\Omega} \supset \Omega$ big enough, $\xi \in \tilde{\Omega}$ and $\eta \in V_\xi$. Let $(\mathcal{F}q)_h \in S^1_{\tilde{h}}(\tilde{\Omega})$ and $q_h \in S^1_{h_q}(\Omega)$. First we compute for every $k = 1, ..., \tilde{N}$

$$(\mathcal{F}q)_h(\xi_k) = \sum_{l=1}^{M} (t^{\eta}[l] - t_0^{\eta}[l]) \int_{\Gamma_l} e_{-(\xi_k+\eta)} \, ds_x.$$

Then we compute

$$q_h(x_j) = \frac{1}{(2\pi)^d} \int_{\tilde{\Omega}} (\mathcal{F}q)_h(\xi) e^{i\xi \cdot x_j} \, d\xi$$

for $j = 1, ..., N_q$.

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The algorithm

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Computation of σ

We extend the boundary data with zero and use the discretization $\{\phi_j^1\}_{j=1}^N \subset S_h^1(\Omega)$. Then we get the equation

$$\tilde{A}_{II}\underline{\sigma}_0^{1/2} = \tilde{f},$$

where

$$\tilde{A}[i,j] = \langle \nabla\phi_j^1, \nabla\phi_i^1 \rangle_{L^2(\Omega)} + \langle q_h\ \phi_j^1, \phi_i^1 \rangle_{L^2(\Omega)} \qquad \forall i,j = 1, ..., N,$$
$$\tilde{f}[i] = -(\tilde{A}_{\Gamma I}\underline{\sigma}_\Gamma^{1/2})[i] \qquad \forall i = 1, ..., N.$$

and

$$\underline{\sigma}^{1/2} = \underline{\sigma}_0^{1/2} + \underline{\sigma}_\Gamma^{1/2}$$

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Summary

Choose $\hat{\Omega}$. Then we go through the following steps for every point ξ of $\hat{\Omega}$: Choose $\eta \in V_\xi$ with $|\eta|$ big and solve

$$\begin{pmatrix} A_{II} & A_{\Gamma I} & \\ A_{I\Gamma} & A_{\Gamma\Gamma} & -M_h^\top \\ & \frac{1}{2}M_h - K_h & V_h \end{pmatrix} \begin{pmatrix} u_I \\ u_\Gamma \\ t^\eta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ f_\eta \end{pmatrix}.$$

and

$$Vt_0^\eta = (\frac{1}{2}I + K)u_\Gamma.$$

Then we compute

$$\mathcal{F}q(\xi) \approx \sum_{i\in I_\Gamma}(t^\eta[i] - t_0^\eta[i]) \int_{\Gamma_i} e_{-(\xi+\eta)}\, dx.$$

Summary

These results we use to approximate

$$q(y) \approx \frac{1}{(2\pi)^d} \int_{\hat{\Omega}} \mathcal{F}q(\xi)\ e^{iy\cdot\xi}\ d\xi.$$

In the end we solve

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Results

The reconstruction problem - The problem

Let $d = 2$ and

$$\Omega = B((0.75, 0.75), 0.25\sqrt{2}).$$

Consider

$$\sigma(x) = \begin{cases} (1 + 10\exp(\frac{1}{|x-x_m|^2 - \frac{8}{9}r^2}))^2 & \text{für } |x - x_m|^2 < \frac{8}{9}r^2, \\ 1 & \text{otherwise,} \end{cases}$$

where $x_m := (0.75, 0.75)$ and $r := 0.25\sqrt{2}$. With this we get

$$S_\sigma g := \sigma \frac{\partial u}{\partial \nu}\Big|_\Gamma = \frac{\partial v_\sigma}{\partial \nu}\Big|_\Gamma = \tilde{S}_q g.$$

since $\sigma|_\Gamma \equiv 1$.

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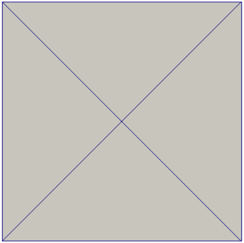
The domain Ω

Consider the domain $\Omega = B((0.75, 0.75), \sqrt{2 \cdot 0.25^2})$. The refinement level 0:



The domain Ω

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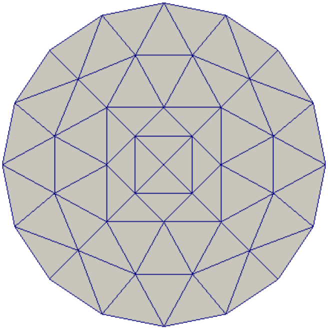
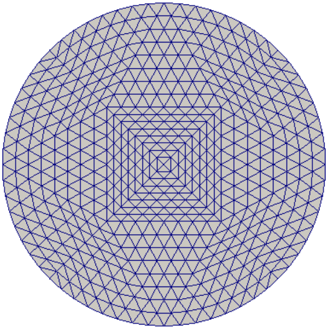


The domain Ω

Consider the domain $\Omega = B((0.75, 0.75), \sqrt{2 \cdot 0.25^2})$. The refinement level 4:

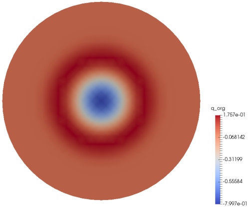
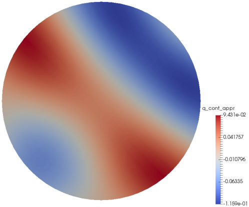


Results for the computation of q

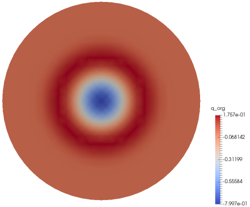
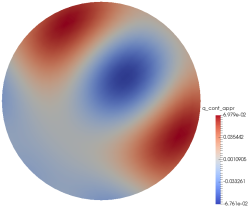
For the fourth refinement step for Ω and different refinement steps for the integration domain $\tilde{\Omega}$, we get

Level	$\|q_h - q\|_{L^2(\Omega)}$
0	0.101697
1	0.087418
2	0.0882527
3	0.0825344
4	0.085346
5	0.0519215

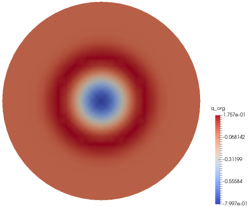
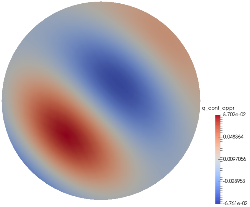
Results for the computation of q

Reflvl	q original	q_h
0		

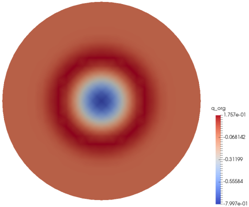
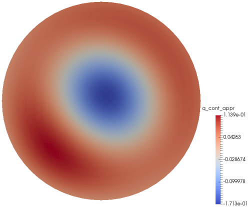
Results for the computation of q

Reflvl	q original	q_h
1		

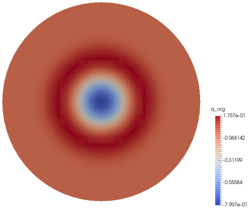
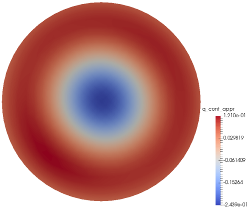
Results for the computation of q

Reflvl	q original	q_h
2		

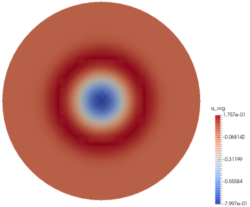
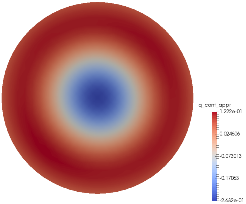
Results for the computation of q

Reflvl	q original	q_h
3		

Results for the computation of q

Reflvl	q original	q_h
4		

Results for the computation of q

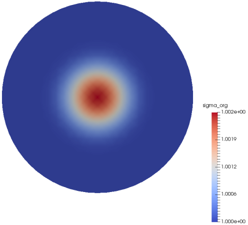
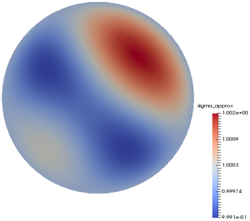
Reflvl	q original	q_h
5		

Results for the computation of σ

For the fourth refinement step for Ω and different refinement steps for the integration domain $\tilde{\Omega}$, we get

Reflvl	$\|\sigma - \sigma_h\|_{L^2(\Omega)}$
0	0.000532299
1	0.000251585
2	0.000441415
3	0.00026305
4	0.000309283
5	0.000149404

Results for the computation of σ

<table>
<tr><th>Reflvl</th><th>σ original</th><th>σ_h</th></tr>
<tr><td>0</td><td>
</td><td>
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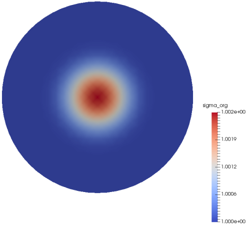
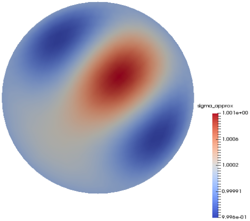
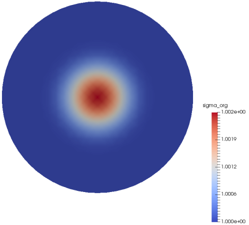
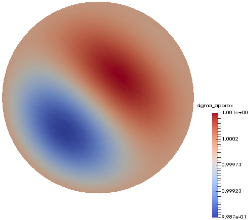
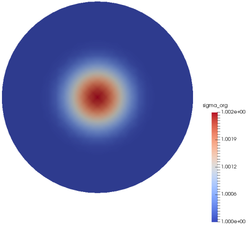
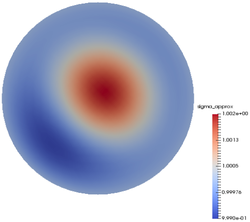
Results for the computation of σ

Reflvl	σ original	σ_h
1		

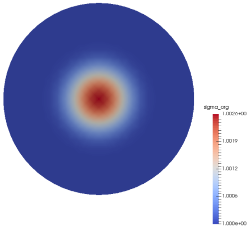
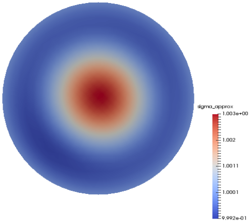
Results for the computation of σ

Reflvl	σ original	σ_h
2		

Results for the computation of σ

Reflvl	σ original	σ_h
3		

Results for the computation of σ

Reflvl	σ original	σ_h
4		

Results for the computation of σ

<table>
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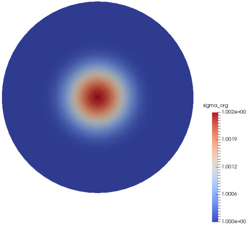
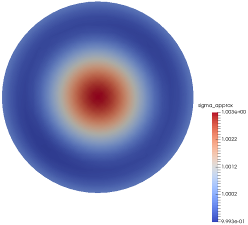
<tr><td>5</td><td>
</td><td>
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Literature

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