Boundary Element Method for Wave Equation

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Motivation

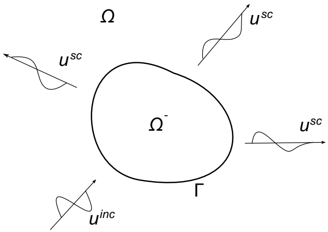
- modelling of wave (acoustic/electromagnetic) propagation has numerous engineering applications
  - nondestructive testing
  - seismology
  - radar
  - ultrasonic imaging
  - tomography
- BEM especially suitable for modelling of wave propagation in an unbounded domain

Wave equation

Scattering problem

$$\begin{cases} \frac{1}{c^2}\frac{\partial^2 u^{sc}}{\partial^2 t}(x,t) - \Delta u^{sc}(x,t) = 0 & \text{in } \Omega \times \mathbb{R}, \\ u^{sc}(x,0) = 0 & \text{in } \Omega, \\ \frac{\partial u^{sc}}{\partial t}(x,0) = 0 & \text{in } \Omega, \\ \mathcal{B}u^{sc}(x,t) = -\mathcal{B}u^{inc}(x,t) & \text{on } \Gamma \times \mathbb{R}_+ \end{cases}$$

- boundary conditions
  - sound-soft scatterer: $\mathcal{B}u = u$
  - sound-hard scatterer: $\mathcal{B}u = \frac{\partial u}{\partial n}$
  - absorbing scatterer: $\mathcal{B}u = \frac{\partial u}{\partial n} - \alpha\frac{\partial u}{\partial t}$



Wave equation

BEM approaches to wave equation

- *Space-time integral equations*
  - use the fundamental solution of the wave equation
  - global in time
  - large system matrix
  - special integration method needed
- *Laplace transform method*
  - solve frequency domain problems and use inverse Laplace/Fourier transform for transform to time domain
- *Time-stepping methods*
  - use implicit scheme for time-discretization and BEM for the solution of resulting elliptic problems in each time step

Fundamental solutions

Lemma

The fundamental solution of the wave equation is given by

$$
\begin{aligned}
G(t,x,y) &= \tfrac{1}{2} H(t - |x-y|) && \text{in 1D,} \\
G(t,x,y) &= \tfrac{1}{2} \tfrac{H(t-|x-y|)}{\sqrt{t^2-|x-y|}} && \text{in 2D,} \\
G(t,x,y) &= \tfrac{1}{4\pi} \tfrac{\delta(t-|x-y|)}{|x-y|} && \text{in 3D.}
\end{aligned}
$$

Representation theorem

Representation formula in 3D

$$
\begin{aligned}
u(t,x) &= \int_0^t \int_\Gamma \frac{\partial}{\partial n(y)} G(t-s, x-y)[u(s,y)] - G(t-s,x-y)[\tfrac{\partial}{\partial n} u(y)] \mathrm{d}\Gamma_y \mathrm{d}s \\
&= \int_0^t \int_\Gamma \frac{\partial}{\partial n(y)} \left(\frac{1}{4\pi|x-y|} \delta(t-s-|x-y|) \right) [u(s,y)] \\
&\quad - \frac{1}{4\pi|x-y|} \delta(t-s-|x-y|)[\tfrac{\partial}{\partial n} u(y)] \mathrm{d}\Gamma_y \mathrm{d}s \\
&= \int_0^t \int_\Gamma \frac{\partial}{\partial n(y)} \frac{1}{4\pi|x-y|} \delta(t-s-|x-y|)[u(s,y)] \\
&\quad - \frac{1}{4\pi|x-y|} \frac{\partial|x-y|}{\partial n(y)} \frac{\partial}{\partial t} \delta(t-s-|x-y|)[u(s,y)] \\
&\quad - \frac{1}{4\pi|x-y|} \delta(t-s-|x-y|)[\tfrac{\partial}{\partial n} u(y)] \mathrm{d}\Gamma_y \mathrm{d}s \\
&= \int_\Gamma \frac{\partial}{\partial n(y)} \frac{1}{4\pi|x-y|} [u(t-|x-y|,y)] - \frac{1}{4\pi|x-y|} \frac{\partial|x-y|}{n(y)} [\tfrac{\partial}{\partial t} u(t-|x-y|)] \\
&\quad - \frac{1}{4\pi|x-y|} [\tfrac{\partial}{\partial n} u(t-|x-y|,y)] \mathrm{d}\Gamma_y
\end{aligned}
$$

Boundary layer potentials

Representation formula in 3D

$$u(t,x) = \int_\Gamma \frac{\partial}{\partial n(y)} \frac{1}{4\pi|x-y|}[u(t-|x-y|,y)] - \frac{1}{4\pi|x-y|}\frac{\partial|x-y|}{n(y)}[\frac{\partial}{\partial t}u(t-|x-y|)]\mathrm{d}\Gamma_y$$
$$-\int_\Gamma \frac{1}{4\pi|x-y|}[\frac{\partial}{\partial n}u(t-|x-y|,y)]\mathrm{d}\Gamma_y = \mathcal{D}([u]) - \mathcal{S}([\partial_n u]), \quad x \in \Omega$$

Let $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^3 \setminus \Gamma$. For $p, \varphi : \mathbb{R}_+ \times \Gamma \to \mathbb{R}$ we define

- single layer potential
  - $(\mathcal{S}([p]))(t,x) := \int_\Gamma \frac{1}{4\pi|x-y|}[p(t-|x-y|,y)]\mathrm{d}\Gamma_y$
- double layer potential
  - $(\mathcal{D}([\varphi]))(t,x) :=$
    $\int_\Gamma \frac{\partial}{\partial n(y)} \frac{1}{4\pi|x-y|}[\varphi(t-|x-y|,y)] - \frac{1}{4\pi|x-y|}\frac{\partial|x-y|}{n(y)}[\frac{\partial}{\partial t}\varphi(t-|x-y|)]\mathrm{d}\,\Gamma_y =$
    $= \frac{1}{4\pi}\int_\Gamma \frac{n(y)(x-y)}{|x-y|}\left(\frac{\varphi(t-|x-y|,y)}{|x-y|^2} + \frac{\dot{\varphi}(t-|x-y|,y)}{|x-y|}\right)\mathrm{d}\,\Gamma_y$

Retarded potential operators

For $x \in \Omega^-$, resp. $x \in \Omega$ going to Γ:

Traces of the potential operators

$$\lim_{\Omega^- \ni x \to \Gamma} (\mathcal{S}(p))(t,x) = \lim_{\Omega \ni x \to \Gamma} (\mathcal{S}(p))(t,x) = Vp(t,x)$$

$$\lim_{\Omega^- \ni x \to \Gamma} \frac{\partial(\mathcal{S}(p))}{\partial n}(t,x) = (I/2 + K)p(t,x)$$

$$\lim_{\Omega \ni x \to \Gamma} \frac{\partial(\mathcal{S}(p))}{\partial n}(t,x) = (-I/2 + K)p(t,x)$$

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Retarded potential operators

Retarded potential operators

$$V p(t,x) := \frac{1}{4\pi} \int_\Gamma \frac{p(\tau,y)}{|x-y|} \mathrm{d}\Gamma_y$$

$$K p(t,x) := \frac{1}{4\pi} \int_\Gamma \frac{n(x)(x-y)}{|x-y|} \left(\frac{p(\tau,y)}{|x-y|^2} + \frac{\dot{p}(\tau,y)}{|x-y|} \right) \mathrm{d}\Gamma_y$$

$$K' \varphi(t,x) := \frac{1}{4\pi} \int_\Gamma \frac{n(y)(x-y)}{|x-y|} \left(\frac{\varphi(\tau,y)}{|x-y|^2} + \frac{\dot{\varphi}(\tau,y)}{|x-y|} \right) \mathrm{d}\Gamma_y$$

$$W \varphi(t,x) := \lim_{\Omega \ni x' \to x} n(x) \nabla_{x'} \left(-\frac{1}{4\pi} \int_\Gamma n(y) \nabla_{x'} \frac{\varphi(t-|x'-y|,y)}{|x'-y|} \right) \mathrm{d}\Gamma_y$$

$$\tau := t - |x-y|$$

- time domain single layer operator
- time domain double layer operator
- time domain adjoint double layer operator
- time domain hypersingular boundary integral operator

Retarded potential boundary integral equations

Direct formulation

Let $u(t,x) = 0$ in Ω^-. Then

$$u(t,x) = \mathcal{D}(u|_\Gamma) - \mathcal{S}(\partial_n u|_\Gamma) \quad \text{in } \mathbb{R}_+ \times \Omega.$$

γ_0^{ex}

$$\gamma_0^{ex} u(t,x) = \gamma_0^{ex}(\mathcal{D}(u|_\Gamma) - \mathcal{S}(\partial_n u|_\Gamma))$$
$$u|_\Gamma = (I/2 + K')(u|_\Gamma) - V(\partial_n u_\Gamma)$$
$$(K' - I/2)(u|_\Gamma) = V(\partial_n u_\Gamma)$$

γ_1^{ex}

$$\gamma_1^{ex} u(t,x) = \gamma_1^{ex}(\mathcal{D}(u|_\Gamma) - \mathcal{S}(\partial_n u|_\Gamma))$$
$$\partial_n u|_\Gamma = W(u|_\Gamma) - (-I/2 + K)(\partial_n u|_\Gamma)$$
$$(K + I/2)(\partial_n u|_\Gamma) = W(u|_\Gamma)$$

Retarded potential boundary integral equations

Indirect formulation

$$u(t,x) = (\mathcal{S}(p))(t,x) \quad \text{in } \mathbb{R}_+ \times \Omega.$$

γ_0^{ex}

$$\gamma_0^{ex} u(t,x) = \gamma_0^{ex}(\mathcal{S}(p))(t,x)$$
$$u|_\Gamma = V(p)$$

γ_1^{ex}

$$\gamma_1^{ex} u(t,x) = \gamma_1^{ex}(\mathcal{S}(p))(t,x)$$
$$\partial_n u|_\Gamma = (-I/2 + K)(p)$$

Retarded potential boundary integral equations

Indirect formulation

$$u(t,x) = (\mathcal{D}(\varphi))(t,x) \quad \text{in } \mathbb{R}_+ \times \Omega.$$

γ_0^{ex}

$$\gamma_0^{ex} u(t,x) = \gamma_0^{ex} (\mathcal{D}(\varphi))(t,x)$$
$$u|_\Gamma = (I/2 + K')(\varphi)$$

γ_1^{ex}

$$\gamma_1^{ex} u(t,x) = \gamma_1^{ex} (\mathcal{D}(\varphi))(t,x)$$
$$\partial_n u|_\Gamma = W(\varphi)$$

Mathematical analysis of RPBIE

- usually done via Laplace transform to frequency domain

$$(\mathcal{L}f)(\omega) = \hat{f} = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega t} f(t)\,\mathrm{d}t$$

- e.g.

$$(\mathcal{L}(Vp))(\omega) = \frac{1}{4\pi}\int_{\Gamma} \frac{\mathrm{e}^{\mathrm{i}\omega|x-y|}}{|x-y|}\hat{p}(y,\omega)\,\mathrm{d}\Gamma_y = \hat{V}_\omega \hat{p}(\omega, x)$$

- RPBIE $\overset{\mathcal{L}}{\to}$ BIE (Helmholtz equation) $\overset{\mathcal{L}^{-1}}{\to}$ RPBIE

Variational formulation

Space-time variational formulation for soft scattering

- indirect formulation using single layer potential for Dirichlet problem

$$V(\phi) = u|_{\Gamma}$$

$$\int_{\Gamma} \frac{\phi(t - |x-y|, y)}{4\pi |x-y|} \, \mathrm{d}\Gamma_y = g(t, x)$$

Weak formulation

Find $\phi \in H^{-1/2,-1/2}([0,T] \times \Gamma) := L^2(0, T, H^{-1/2}(\Gamma)) + H^{-1/2}(0, T, L^2(\Gamma))$ such that

$$\int_0^T \int_{\Gamma} \int_{\Gamma} \frac{\dot{\phi}(t - |x-y|, y)\xi(t, x)}{4\pi |x-y|} \, \mathrm{d}\Gamma_y \, \mathrm{d}\Gamma_x \, \mathrm{d}t = \int_0^T \int_{\Gamma} \dot{g}(x, t)\xi(x, t) \, \mathrm{d}\Gamma_x \, \mathrm{d}t$$

holds for all ξ.

Space-time Galerkin discretization

Discretization

$$\phi_{\text{Galerkin}} = \sum_{i=1}^{N} \sum_{i=1}^{M} \alpha_i^j \varphi_j(x) b_i(t), \quad (x,t) \in \mathit{\Gamma}$$

- $\{b_i\}_{i=1}^{N} \ldots$ basis functions in time (with compact supports)
- $\{\varphi_j\}_{j=1}^{M} \ldots$ basis functions in space (with compact supports)
- $\alpha_i^j \ldots$ unknown coefficients

Space-time Galerkin discretization

Galerkin discretization

Find $\alpha_i^j, i = 1, \ldots, N, j = 1, \ldots, M$ such that

$$\int_0^T \int_\Gamma \int_\Gamma \sum_{i=1}^{N} \sum_{j=1}^{M} \frac{\alpha_i^j \varphi_j(y) \dot{b}_i(t - |x-y|) \varphi_l(x) b_k(t)}{4\pi |x-y|} \, \mathrm{d}\Gamma_y \, \mathrm{d}\Gamma_x \, \mathrm{d}t$$
$$= \int_0^T \int_\Gamma \dot{g}(x,t) \varphi_l(x) b_k(t) \mathrm{d}\Gamma_x \, \mathrm{d}t$$

for $k = 1, \ldots, N, l = 1, \ldots, M$.

$$\psi_{i,k}(r) := \int_0^T \frac{\dot{b}_i(t-r) b_k(t)}{4\pi r} \, \mathrm{d}t$$

$$A_{j,l}^{i,k} := \int_\Gamma \int_\Gamma \varphi_j(y) \varphi_l(x) \psi_{i,k}(|x-y|) \, \mathrm{d}\Gamma_y \, \mathrm{d}\Gamma_x$$
$$= \int_{\mathrm{supp}(\varphi_l)} \int_{\mathrm{supp}(\varphi_j)} \varphi_j(y) \varphi_l(x) \psi_{i,k}(|x-y|) \, \mathrm{d}\Gamma_y \, \mathrm{d}\Gamma_x$$

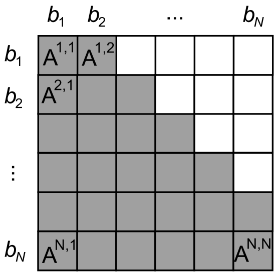
Space-time Galerkin discretization

Galerkin discretization

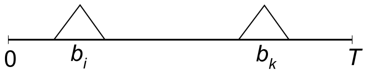
Find $\alpha_i^j, i = 1, \ldots, N, j = 1, \ldots, M$ such that

$$\sum_{i=1}^{N} \sum_{j=1}^{M} A_{j,l}^{i,k} \alpha_i^j = \underbrace{\int_0^T \int_\Gamma \dot{g}(x,t) \varphi_l(x) b_k(t) \mathrm{d}\Gamma_x \, \mathrm{d}t}_{=: g_l^k}$$

for $k = 1, \ldots, N, l = 1, \ldots, M$.



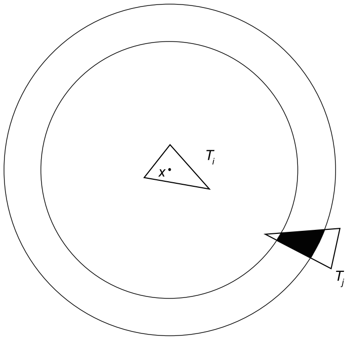
Temporal basis functions



Integration problem

How to efficiently evaluate $A_{j,l}^{i,k}$?

- $\psi_{i,k}(r) := \int_0^T \frac{\dot{b}_i(t-r)b_k(t)}{4\pi r}\,\mathrm{d}t$ is non-zero only for $r = |x - y|$ such that $\mathrm{supp}(\dot{b}_i(t-r)) \cap \mathrm{supp}(b_j(t)) \neq \emptyset$

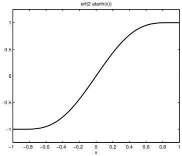


Temporal basis functions

- construction of infinitely smooth temporal basis functions using partition of unity method (PUM), [Sauter, Veit]

Let us start with the C^∞ function

$$f(t) := \begin{cases} \operatorname{erf}(2\operatorname{arctanh}(t)), \text{for } |t| < 1, \\ -1, \text{for } t \leq -1, \\ 1, \text{for } t \geq 1. \end{cases}$$



Then

$$h_{a,b}(t) := \frac{1}{2} f\left(2\frac{t-a}{b-a} - 1\right) + \frac{1}{2},$$

and

$$\rho_{a,b,c}(t) := \begin{cases} h_{a,b}(t,), \text{for } t \leq b, \\ 1 - h_{b,c}(t), \text{for } t \geq b. \end{cases}$$

Temporal basis functions

Partition of unity functions

Let $\Theta = \langle 0, T \rangle$ and $0 = t_0 < t_1 < t_2 < \ldots < t_{N-2} < t_{N-1} = T, \tau_i := \langle t_{i-1}, t_i \rangle$. Let $\Theta_1 := \tau_1, \Theta_1 := \tau_1, \Theta_i := \tau_{i-1} \cup \tau_i, i = 2, \ldots N-2, \Theta_N := \tau_{N-1}$. Then a smooth partition of unity subordinate to the cover $\{\Theta_i\}$ is defined as

$$
\begin{aligned}
\varphi_1(t) &:= 1 - h_{t_0,t_1}(t), \\
\varphi_i(t) &:= \rho_{t_{i-2},t_{i-1},t_i}(t), \quad \text{for } i = 2, \ldots, N-1, \\
\varphi_N(t) &:= h_{t_{N-2},t_{N-1}}(t).
\end{aligned}
$$

Temporal basis functions

The temporal basis functions are defined as

$$
\begin{aligned}
b_1(t) &:= \varphi_1(t) t^2, \\
b_i(t) &:= \varphi_i(t), \quad \text{for } i = 2, \ldots, N-1, \\
b_N(t) &:= \varphi_N(t).
\end{aligned}
$$

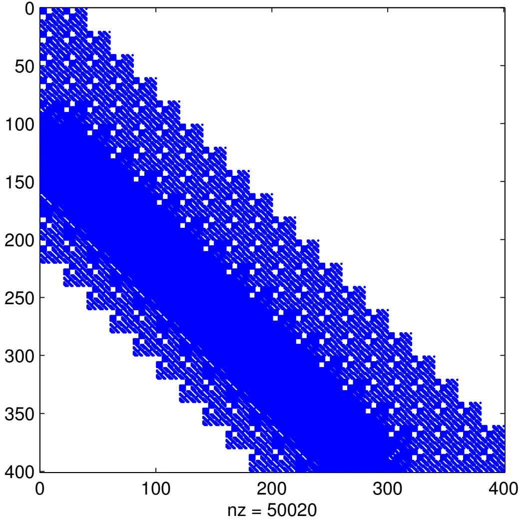
Implementation remarks

Algorithm 1 System matrix assembly

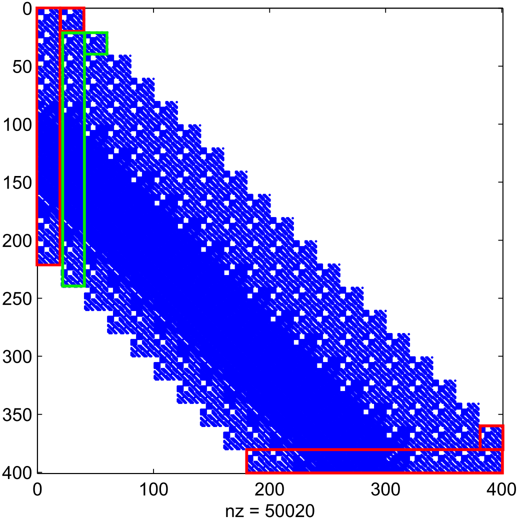
Require: A triangulation $\{\tau_i : 1 \leq i \leq M\}$ of Γ, number of time-steps N, time derivative g of RHS

for $k = 1$ to N **do**
  $g_k \leftarrow \left(\int_0^T \int_\Gamma \dot{g}(x,t) \varphi_l(x) b_k(t) \, \mathrm{d}\Gamma_x \mathrm{d}t \right)_{l=1}^{M} \in \mathbb{R}^M$
  for $i = 1$ to N **do**
    if $\min \operatorname{supp} b_i \geq \max \operatorname{supp} b_k$ **then**
      $A^{k,i} \leftarrow 0 \in \mathbb{R}^{M \times M}$
    else
      for $j, l = 1$ to M **do**
        $A^{k,i}_{j,l} \leftarrow \int_0^T \int_\Gamma \int_\Gamma \varphi_j(y) \varphi_l(x) \psi(r) \, \mathrm{d}\Gamma_y \, \mathrm{d}\Gamma_x$
      end for
    end if
  end for
end for

Matrix structure



Matrix structure



Solving the system

What kind of solver should we use?

- iterative (GMRES, BiCGStab)
  - would be ideal because of low memory requirements
  - missing suitable preconditioners
- direct (PARDISO, SuperLU, MUMPS)
  - high memory requirements

Current work

- optimizing and parallelizing system matrix assembly
- tests of direct solvers
  - MUMPS - 5120 elements, 25 time steps - approx. 15 min. on ANSELM
- MPI parallelization necessary
- matrix approximation?
- preconditioners?

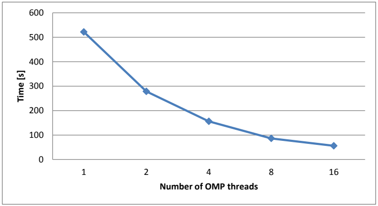


Figure : Assemly of hypersingular operator matrix

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Thank you for your attention!