Space–time finite element methods for parabolic problems

Olaf Steinbach

Institut für Angewandte Mathematik, TU Graz

http://www.applied.math.tugraz.at

based on joint work with Marco Zank

- construction of space–time adaptive solutions of partial differential equations
- unified analysis of space–time finite and boundary element methods
- solvability of time dependent partial differential equations in Bochner spaces and in anisotropic Sobolev spaces
- a posteriori error estimates and space–time adaptivity
- parallel iterative solution methods and preconditioners
- stable space–time variational formulations for time–dependent partial differential equations and finite element methods

Dirichlet boundary value problem for the heat equation

$$\begin{aligned} \alpha\partial_t u(x,t) - \Delta_x u(x,t) &= f(x,t) && \text{for } (x,t) \in Q := \Omega \times (0,T), \\ u(x,t) &= 0 && \text{for } (x,t) \in \Sigma := \Gamma \times (0,T), \\ u(x,0) &= 0 && \text{for } x \in \Omega. \end{aligned}$$

Find $u \in L^2(0,T;H^1_0(\Omega)) \cap H^1_{0,}(0,T;H^{-1}(\Omega))$ such that

$$\int_0^T \int_\Omega \Big[\alpha\partial_t u(x,t)v(x,t) + \nabla_x u(x,t)\cdot\nabla_x v(x,t)\Big] dxdt = \int_0^T \int_\Omega f(x,t)v(x,t)dxdt$$

is satisfied for all $v \in L^2(0,T;H^1_0(\Omega))$.

Bilinear form

$$\begin{aligned} a(u,v) &= \int_0^T \int_\Omega \Big[\alpha\partial_t u(x,t)v(x,t) + \nabla_x u(x,t)\cdot\nabla_x v(x,t)\Big] dxdt \\ &= \int_0^T \int_\Omega \Big[\alpha\partial_t u(x,t) - \Delta_x u(x,t)\Big]\, v(x,t)\, dxdt \end{aligned}$$

Quasi–static Dirichlet problem for $u \in L^2(0,T;H^1_0(\Omega)) \cap H^1_{0,}(0,T;H^{-1}(\Omega))$

$$\begin{array}{rcll} -\Delta_x w(x,t) & = & \alpha \partial_t u(x,t) - \Delta_x u(x,t) & \text{for } (x,t) \in Q, \\ w(x,t) & = & 0 & \text{for } (x,t) \in \Sigma. \end{array}$$

Find $w \in L^2(0,T;H^1_0(\Omega))$ such that

$$\int_0^T \int_\Omega \nabla_x w(x,t) \cdot \nabla_x v(x,t)\, dxdt = \int_0^T \int_\Omega \Big[\alpha \partial_t u(x,t) - \Delta_x u(x,t) \Big]\, v(x,t)\, dxdt$$

is satisfied for all $v \in L^2(0,T;H^1_0(\Omega))$.

Lemma

$$\|\alpha \partial_t u - \Delta_x u\|^2_{L^2(0,T;H^{-1}(\Omega))} = \|w\|^2_{L^2(0,T;H^1_0(\Omega))} = a(u,w)$$

Corollary

$$\|\alpha \partial_t u - \Delta_x u\|_{L^2(0,T;H^{-1}(\Omega))} \leq \sup_{0 \neq v \in L^2(0,T;H^1_0(\Omega))} \frac{a(u,v)}{\|v\|_{L^2(0,T;H^1_0(\Omega))}}$$

Norm

$$\begin{aligned}\|\alpha\partial_t u - \Delta_x u\|^2_{L^2(0,T;H^{-1}(\Omega))} &= \|w\|^2_{L^2(0,T;H^1_0(\Omega))} = a(u,w) \\ &= \int_0^T \int_\Omega \Big[\alpha\partial_t u\, w + \nabla_x u \cdot \nabla_x w\Big] dxdt \\ &\le \Big[\|\alpha\partial_t u\|_{L^2(0,T;H^{-1}(\Omega))} + \|u\|_{L^2(0,T;H^1_0(\Omega))}\Big] \|w\|_{L^2(0,T;H^1_0(\Omega))} \\ &\le \sqrt{2}\Big[\|\alpha\partial_t u\|^2_{L^2(0,T;H^{-1}(\Omega))} + \|u\|^2_{L^2(0,T;H^1_0(\Omega))}\Big]^{1/2} \|w\|_{L^2(0,T;H^1_0(\Omega))}\end{aligned}$$

i.e.

$$\|\alpha\partial_t u - \Delta_x u\|^2_{L^2(0,T;H^{-1}(\Omega))} \le 2\Big[\|\alpha\partial_t u\|^2_{L^2(0,T;H^{-1}(\Omega))} + \|u\|^2_{L^2(0,T;H^1_0(\Omega))}\Big]$$

Norm

$$\begin{aligned}
\|\alpha\partial_t u - \Delta_x u\|^2_{L^2(0,T;H^{-1}(\Omega))} &= a(u,w) = a(u,u) + a(u,w-u) \\
&= \int_0^T \int_\Omega \Big[\alpha\partial_t u\, u + \nabla_x u \cdot \nabla_x u\Big] dxdt + \int_0^T \int_\Omega \nabla_x w \cdot \nabla_x (w-u) dxdt \\
&= \frac{\alpha}{2}\|u(T)\|^2_{L^2(\Omega)} + \|\nabla_x u\|^2_{L^2(Q)} + \|\nabla_x(w-u)\|^2_{L^2(Q)} + \int_0^T\int_\Omega \nabla_x u \cdot \nabla_x (w-u) dxdt \\
&\geq \|\nabla_x u\|^2_{L^2(Q)} + \|\nabla_x(w-u)\|^2_{L^2(Q)} - \|\nabla_x u\|^2_{L^2(Q)}\|\nabla_x(w-u)\|^2_{L^2(Q)} \\
&\geq \frac{1}{2}\Big[\|\nabla_x u\|^2_{L^2(Q)} + \|\nabla_x(w-u)\|^2_{L^2(Q)}\Big] \\
&= \frac{1}{2}\Big[\|\alpha\partial_t u\|^2_{L^2(0,T;H^{-1}(\Omega))} + \|u\|^2_{L^2(0,T;H^1_0(\Omega))}\Big]
\end{aligned}$$

Norm equivalence

$$\frac{1}{2}\Big[\|\alpha\partial_t u\|^2_{L^2(0,T;H^{-1}(\Omega))} + \|u\|^2_{L^2(0,T;H^1_0(\Omega))}\Big] \le \|\alpha\partial_t u - \Delta_x u\|^2_{L^2(0,T;H^{-1}(\Omega))}$$
$$\le 2\Big[\|\alpha\partial_t u\|^2_{L^2(0,T;H^{-1}(\Omega))} + \|u\|^2_{L^2(0,T;H^1_0(\Omega))}\Big]$$

Lemma

$$\frac{1}{\sqrt{2}}\sqrt{\|\alpha\partial_t u\|^2_{L^2(0,T;H^{-1}(\Omega))} + \|u\|^2_{L^2(0,T;H^1_0(\Omega))}} \le \sup_{0\neq v\in L^2(0,T;H^1_0(\Omega))} \frac{a(u,v)}{\|v\|_{L^2(0,T;H^1_0(\Omega))}}$$

[Schwab, Stevenson 2009; Urban, Patera 2014; Andreev 2013; Mollet 2014; OS 2015; ...]

For $v \in L^2(0,T;H^1_0(\Omega))$ define

$$\widetilde{u}(x,t) := \int_0^t v(x,s)\,ds \quad \text{for } x \in \Omega,\ t \in [0,T]$$

and compute

$$\begin{aligned} a(\widetilde{u},v) &= \int_0^T \int_\Omega \Big[\alpha \partial_t \widetilde{u}(x,t) v(x,t) + \nabla_x \widetilde{u}(x,t) \cdot \nabla_x v(x,t)\Big]\,dx\,dt \\ &= \int_0^T \int_\Omega \Big[\alpha [\partial_t \widetilde{u}(x,t)]^2 + \nabla_x \widetilde{u}(x,t) \cdot \nabla_x \partial_t \widetilde{u}(x,t)\Big]\,dx\,dt \\ &= \alpha\, \|\partial_t \widetilde{u}\|^2_{L^2(Q)} + \frac{1}{2}\, \|\nabla_x \widetilde{u}(T)\|^2_{L^2(\Omega)} > 0 \end{aligned}$$

Find $u \in L^2(0,T;H^1_0(\Omega)) \cap H^1_{0,}(0,T;H^{-1}(\Omega))$ such that

$$\int_0^T \int_\Omega \Big[\alpha \partial_t u(x,t) v(x,t) + \nabla_x u(x,t) \cdot \nabla_x v(x,t)\Big] dxdt = \int_0^T \int_\Omega f(x,t) v(x,t) dxdt$$

is satisfied for all $v \in L^2(0,T;H^1_0(\Omega))$.

Theorem

The above variational problem admits a unique solution u.

Finite element spaces

$$X_h \subset X \quad := \quad L^2(0,T;H^1_0(\Omega)) \cap H^1_{0,}(0,T;H^{-1}(\Omega))$$
$$Y_h \subset Y \quad := \quad L^2(0,T;H^1_0(\Omega))$$

Assumption

$$X_h \subset Y_h$$

Consider

$$X_h = Y_h$$

Find $u_h \in X_h$ such that

$$\int_0^T \int_\Omega \Big[\alpha \partial_t u_h(x,t) v_h(x,t) + \nabla_x u_h(x,t) \cdot \nabla_x v_h(x,t) \Big] dxdt = \int_0^T \int_\Omega f(x,t) v_h(x,t) dxdt$$

is satisfied for all $v_h \in Y_h$.

Find $w \in Y = L^2(0, T; H^1_0(\Omega))$ such that

$$\int_0^T \int_\Omega \nabla_x w(x,t) \cdot \nabla_x v(x,t)\, dx\, dt = \int_0^T \alpha \partial_t u(x,t) v(x,t)\, dx\, dt$$

is satisfied for all $v \in Y$.

Find $w_h \in Y_h$ such that

$$\int_0^T \int_\Omega \nabla_x w_h(x,t) \cdot \nabla_x v_h(x,t)\, dx\, dt = \int_0^T \alpha \partial_t u(x,t) v_h(x,t)\, dx\, dt$$

is satisfied for all $v_h \in Y_h$.

Norms

$$\begin{aligned} \|u\|^2_X &= \|\alpha \partial_t u\|^2_{L^2(0,T;H^{-1}(\Omega))} + \|u\|^2_{L^2(0,T;H^1_0(\Omega))} \\ &= \|w\|^2_{L^2(0,T;H^1_0(\Omega))} + \|u\|^2_{L^2(0,T;H^1_0(\Omega))} \\ \|u\|^2_{X_h} &= \|w_h\|^2_{L^2(0,T;H^1_0(\Omega))} + \|u\|^2_{L^2(0,T;H^1_0(\Omega))} \end{aligned}$$

Theorem

$$\frac{1}{2\sqrt{2}}\,\|u_h\|_{X_h} \leq \sup_{0\neq v_h\in Y_h} \frac{a(u_h,v_h)}{\|v_h\|_{L^2(0,T;H^1_0(\Omega))}} \quad \text{for all } u_h \in X_h$$

Proof: Due to $X_h \subset Y_h$ we have $u_h + w_h \in Y_h$ satisfying

$$\|u_h+w_h\|^2_{L^2(0,T;H^1_0(\Omega))} \leq 2\left[\|u_h\|^2_{L^2(0,T;H^1_0(\Omega))} + \|w_h\|^2_{L^2(0,T;H^1_0(\Omega))}\right] = 2\,\|u_h\|^2_{X_h}\,.$$

Then,

$$\begin{aligned} a(u_h,u_h+w_h) &= a(u_h,u_h)+a(u_h,w_h) \\ a(u_h,u_h) &= \frac{\alpha}{2}\,\|u_h(T)\|^2_{L^2(\Omega)} + \|u_h\|^2_{L^2(0,T;H^1_0(\Omega))} \end{aligned}$$

$$\begin{aligned} a(u_h, w_h) &= \int_0^T \int_\Omega \Big[\alpha \partial_t u_h(x,t) w_h(x,t) + \nabla_x u_h(x,t) \cdot \nabla_x w_h(x,t)\Big]\, dx\, dt \\ &= \int_0^T \int_\Omega \nabla_x w_h(x,t) \cdot \nabla_x w_h(x,t)\, dx\, dt + \int_0^T \int_\Omega \nabla_x u_h(x,t) \cdot \nabla_x w_h(x,t)\, dx\, dt \\ &\geq \|w_h\|^2_{L^2(0,T;H^1_0(\Omega))} - \|u_h\|_{L^2(0,T;H^1_0(\Omega))} \|w_h\|_{L^2(0,T;H^1_0(\Omega))} \end{aligned}$$

$$\begin{aligned} a(u_h, u_h + w_h) &\geq \\ &\geq \|u_h\|^2_{L^2(0,T;H^1_0(\Omega))} + \|w_h\|^2_{L^2(0,T;H^1_0(\Omega))} - \|u_h\|_{L^2(0,T;H^1_0(\Omega))} \|w_h\|_{L^2(0,T;H^1_0(\Omega))} \\ &\geq \frac{1}{2} \Big[\|u_h\|^2_{L^2(0,T;H^1_0(\Omega))} + \|w_h\|^2_{L^2(0,T;H^1_0(\Omega))} \Big] \\ &= \frac{1}{2} \|u_h\|^2_{X_h} \geq \frac{1}{2\sqrt{2}} \|u_h\|_{X_h} \|u_h + w_h\|_{L^2(0,T;H^1_0(\Omega))} \end{aligned}$$

Theorem

$$\|u - u_h\|_{X_h} \leq 5 \inf_{z_h \in X_h} \|u - z_h\|_X$$

Finite element spaces (pw linear, continuous)

$$X_h = Y_h = S_h^1(Q_h) \cap X$$

Theorem

$$\|u - u_h\|_{L^2(0,T;H_0^1(\Omega))} \leq c\, h\, |u|_{H^2(Q)}$$

Find $u \in L^2(0,T;H^1_0(\Omega)) \cap H^1_{0,}(0,T;H^{-1}(\Omega))$ such that

$$\int_0^T \int_\Omega \Big[\alpha \partial_t u(x,t) v(x,t) + \nabla_x u(x,t) \cdot \nabla_x v(x,t) \Big] dx dt = \int_0^T \int_\Omega f(x,t) v(x,t) dx dt$$

is satisfied for all $v \in L^2(0,T;H^1_0(\Omega))$.

Operator

$$\mathcal{L} : L^2(0,T;H^1_0(\Omega)) \cap H^1_{0,}(0,T;H^{-1}(\Omega)) \to [L^2(0,T;H^1_0(\Omega))]'$$

Adjoint variational formulation to find $u \in L^2(0,T;H^1_0(\Omega))$ such that

$$\int_0^T\int_\Omega \Big[-u(x,t)\alpha\partial_t v(x,t) + \nabla_x u(x,t)\cdot\nabla_x v(x,t)\Big]dxdt = \int_0^T\int_\Omega f(x,t)v(x,t)dxdt$$

is satisfied for all $v \in L^2(0,T;H^1_0(\Omega)) \cap H^1_{,0}(0,T;H^{-1}(\Omega))$.

Operator adjoint formulation

$$\mathcal{L} : L^2(0,T;H^1_0(\Omega)) \to [L^2(0,T;H^1_0(\Omega)) \cap H^1_{,0}(0,T;H^{-1}(\Omega))]'$$

Operator primal formulation

$$\mathcal{L} : L^2(0,T;H^1_0(\Omega)) \cap H^1_{0,}(0,T;H^{-1}(\Omega)) \to [L^2(0,T;H^1_0(\Omega))]'$$

Can we define

$$\mathcal{L} : L^2(0,T;H^1_0(\Omega)) \cap H^{1/2}_{0,}(0,T;L^2(\Omega)) \to [L^2(0,T;H^1_0(\Omega)) \cap H^{1/2}_{,0}(0,T;L^2(\Omega))]'$$

Model problem

$$u'(t) = f(t) \quad \text{for } t \in (0,T), \quad u(0) = 0$$

Primal variational formulation: Find $u \in H^1_{0,}(0,T)$ such that

$$a(u,v) = \int_0^T u'(t)v(t)\,dt = \int_0^T f(t)v(t)\,dt \quad \text{for all } v \in L^2(0,T)$$

Lemma

$$a(u,v) \le \|u'\|_{L^2(0,T)}\|v\|_{L^2(0,T)} \quad \text{for all } u \in H^1_{0,}(0,T), v \in L^2(0,T)$$

$$\|u'\|_{L^2(0,T)} \le \sup_{0 \ne v \in L^2(0,T)} \frac{a(u,v)}{\|v\|_{L^2(0,T)}} \quad \text{for all } u \in H^1_{0,}(0,T)$$

$$\|v\|_{L^2(0,T)} \le \sup_{0 \ne u \in H^1_{0,}(0,T)} \frac{a(u,v)}{\|u'\|_{L^2(0,T)}} \quad \text{for all } v \in L^2(0,T)$$

Primal variational formulation: Find $u \in H^1_{0,}(0,T)$ such that

$$a(u,v) = \int_0^T u'(t)v(t)\,dt = \int_0^T f(t)v(t)\,dt \quad \text{for all } v \in L^2(0,T)$$

Solution operator

$$B_1 : H^1_{0,}(0,T) \to L^2(0,T)$$

Dual variational formulation: Find $u \in L^2(0,T)$ such that

$$\int_0^T u(t)v'(t)\,dt = -\int_0^T f(t)v(t)\,dt \quad \text{for all } v \in H^1_{,0}(0,T)$$

Solution operator

$$B_0 : L^2(0,T) \to [H^1_{,0}(0,T)]'$$

Question

$$B_s : [L^2(0,T), H^1_{0,}(0,T)]_s \to [[H^1_{,0}(0,T)]', L^2(0,T)]_s \quad \text{for } s \in (0,1), s = \frac{1}{2}$$

Related work: [Fontes 1996; Langer, Wolfmayr 2013; Larsson, Schwab 2015]

Solution operator

$$B_1 : H^1_{0,}(0,T) \to L^2(0,T)$$

Adjoint operator

$$B_1' : L^2(0,T) \to [H^1_{0,}(0,T)]'$$

$$\langle u, B_1' v\rangle_{L^2(0,T)} = \langle B_1 u, v\rangle_{L^2(0,T)} \quad \text{for all } u \in H^1_{0,}(0,T), v \in L^2(0,T)$$

Define

$$A := B_1' B_1 : H^1_{0,}(0,T) \to [H^1_{0,}(0,T)]'$$

Eigenvalue problem

$$Au = \lambda u$$

$$\langle Au, v\rangle_{L^2(0,T)} = \langle B_1 u, B_1 v\rangle_{L^2(0,T)} = \int_0^T \partial_t u(t) \partial_t v(t)\, dt = \lambda \int_0^T u(t) v(t)\, dt$$

$$-u''(t) = \lambda u(t) \quad \text{for } t \in (0,T), \quad u(0) = 0, \quad u'(T) = 0$$

$$v_k(t) = \sin\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}, \quad \lambda_k = \frac{1}{T^2}\left(\frac{\pi}{2} + k\pi\right)^2, \quad k = 0,1,2,3,\ldots$$

$u \in H^1_{0,}(0,T)$

$$u(t) = \sum_{k=0}^{\infty} u_k \sin\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}, \quad u_k = \frac{2}{T}\int_0^T u(t) \sin\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}\, dt$$

Time derivative

$$(B_1 u)(t) = u'(t) = \frac{1}{T}\sum_{k=0}^{\infty} u_k \left(\frac{\pi}{2} + k\pi\right) \cos\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}$$

$$\begin{aligned}
\langle B_1 u, v\rangle_{L^2(0,T)} &= \frac{1}{T}\int_0^T \sum_{k=0}^{\infty} u_k \left(\frac{\pi}{2} + k\pi\right) \cos\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}\, v(t)\, dt \\
&= \frac{1}{T}\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} u_k u_\ell \left(\frac{\pi}{2} + k\pi\right) \int_0^T \cos\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T} \cos\left(\frac{\pi}{2} + \ell\pi\right)\frac{t}{T}\, dt \\
&= \frac{1}{2}\sum_{k=0}^{\infty} u_k^2 \left(\frac{\pi}{2} + k\pi\right) = \|u\|^2_{H^{1/2}_{0,}(0,T)}
\end{aligned}$$

if we choose

$$v(t) = \sum_{\ell=0}^{\infty} u_\ell \cos\left(\frac{\pi}{2} + \ell\pi\right)\frac{t}{T},$$

Transformation operator

$$(\mathcal{H}_T u)(t) = \sum_{k=0}^{\infty} u_k \cos\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}, \quad u(t) = \sum_{k=0}^{\infty} u_k \sin\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T},$$

Lemma

$$\mathcal{H}_T : H^s_{0,}(0,T) \to H^s_{,0}(0,T), \quad \|\mathcal{H}_T u\|_{H^s_{,0}(0,T)} = \|u\|_{H^s_{0,}(0,T)}$$

Inverse transformation operator

$$(\mathcal{H}_T^{-1} v)(t) = \sum_{k=0}^{\infty} v_k \sin\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}, \quad v(t) = \sum_{k=0}^{\infty} v_k \cos\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}$$

Lemma

$$\partial_t \mathcal{H}_T u = -\mathcal{H}_T^{-1} \partial_t u$$

Proof: Consider first $\varphi \in C^\infty([0,T])$ with $\varphi(0) = 0$:

$$\varphi(t) = \sum_{k=0}^\infty \varphi_k \sin\left(\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}\right), \; \varphi_k = \frac{2}{T}\int_0^T \varphi(t) \sin\left(\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}\right)\, dt$$

$$(\mathcal{H}_T\varphi)(t) = \sum_{k=0}^\infty \varphi_k \cos\left(\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}\right)$$

$$\partial_t(\mathcal{H}_T\varphi)(t) = -\frac{1}{T}\sum_{k=0}^\infty \varphi_k \left(\frac{\pi}{2} + k\pi\right) \sin\left(\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}\right).$$

$$\partial_t\varphi(t) = \frac{1}{T}\sum_{k=0}^\infty \varphi_k \left(\frac{\pi}{2} + k\pi\right) \cos\left(\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}\right)$$

$$(\mathcal{H}_T^{-1}\partial_t\varphi)(t) = \frac{1}{T}\sum_{k=0}^\infty \varphi_k \left(\frac{\pi}{2} + k\pi\right) \sin\left(\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}\right),$$

i.e.

$$\partial_t\mathcal{H}_T\varphi = -\mathcal{H}_T^{-1}\partial_t\varphi \quad \text{for all } \varphi \in C^\infty([0,T]) \text{ with } \varphi(0) = 0.$$

Lemma: For $u \in H^{1/2}_{0,}(0,T)$ and $w \in H^{1/2}_{,0}(0,T)$ we have

$$\langle \mathcal{H}_T u, w\rangle_{L^2(0,T)} = \langle u, \mathcal{H}_T^{-1} w\rangle_{L^2(0,T)}$$

Proof:

$$u(t) = \sum_{k=0}^\infty u_k \sin\left(\left(\frac{\pi}{2}+k\pi\right)\frac{t}{T}\right), \quad w(t) = \sum_{\ell=0}^\infty \overline{w}_\ell \cos\left(\left(\frac{\pi}{2}+\ell\pi\right)\frac{t}{T}\right),$$

$$\langle \mathcal{H}_T u, w\rangle_{L^2(0,T)} = \int_0^T (\mathcal{H}_T u)(t) w(t)\, dt$$

$$= \sum_{k=0}^\infty \sum_{\ell=0}^\infty u_k \overline{w}_\ell \int_0^T \cos\left(\left(\frac{\pi}{2}+k\pi\right)\frac{t}{T}\right) \cos\left(\left(\frac{\pi}{2}+\ell\pi\right)\frac{t}{T}\right) dt$$

$$= \frac{T}{2}\sum_{k=0}^\infty u_k \overline{w}_k = \sum_{k=0}^\infty \sum_{\ell=0}^\infty u_k \overline{w}_\ell \int_0^T \sin\left(\left(\frac{\pi}{2}+k\pi\right)\frac{t}{T}\right) \sin\left(\left(\frac{\pi}{2}+\ell\pi\right)\frac{t}{T}\right) dt$$

$$= \int_0^T u(t)(\mathcal{H}_T^{-1} w)(t)\, dt = \langle u, \mathcal{H}_T^{-1} w\rangle_{L^2(0,T)}.$$

Lemma

$$\partial_t \mathcal{H}_T u = -\mathcal{H}_T^{-1} \partial_t u$$

Lemma

$$\langle \mathcal{H}_T u, w \rangle_{L^2(0,T)} = \langle u, \mathcal{H}_T^{-1} w \rangle_{L^2(0,T)}$$

Corollary

$$\langle \partial_t u, \mathcal{H}_T v \rangle_{(0,T)} = \langle \mathcal{H}_T u, \partial_t v \rangle_{(0,T)}$$

Lemma

$$\langle v, \mathcal{H}_T v\rangle_{L^2(0,T)} > 0 \quad \text{for all } 0 \neq v \in H^{1/2}_{0,}(0,T)$$

Proof:

$$\begin{aligned}
\langle v, \mathcal{H}_T v\rangle_{L^2(0,T)} &= \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} v_k v_\ell \int_0^T \sin\left(\left(\frac{\pi}{2}+k\pi\right)\frac{t}{T}\right)\cos\left(\left(\frac{\pi}{2}+\ell\pi\right)\frac{t}{T}\right) dt \\
&= \frac{T}{2\pi}\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} v_k v_\ell \frac{1}{k+\ell+1}\left[1-(-1)^{k+\ell+1}\right] \\
&= \frac{T}{\pi}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left[\frac{v_{2i}v_{2j}}{2i+2j+1}+\frac{v_{2i+1}v_{2j+1}}{2i+2j+3}\right] \\
&= \frac{T}{\pi}\left[\int_0^1\left(\sum_{i=0}^{\infty} v_{2i}x^{2i}\right)^2 dx + \int_0^1\left(\sum_{i=0}^{\infty} v_{2i+1}x^{2i+1}\right)^2 dx\right] \\
&> 0.
\end{aligned}$$

Lemma

$$(\mathcal{H}_T u)(t) = \frac{1}{2T} \int_0^T \left[\frac{1}{\sin\left(\frac{\pi}{2}\frac{s-t}{T}\right)} + \frac{1}{\sin\left(\frac{\pi}{2}\frac{s+t}{T}\right)} \right] u(s)\, ds$$

Corollary: For $T \to \infty$

$$(\mathcal{H}_\infty u)(t) = \frac{1}{\pi} \int_0^\infty \frac{u(s)}{s-t} \frac{2s}{s+t}\, ds$$

$\to$ Hilbert transformation

$\to$ Construction of optimal test functions [Demkowicz, Gopalakrishnan 2011]

Model problem

$$u'(t) = f(t) \quad \text{for } t \in (0,T), \quad u(0) = 0$$

Variational formulation: Find $u \in H^{1/2}_{0,}(0,T)$ such that

$$\langle \partial_t u, v \rangle_{(0,T)} = \langle f, v \rangle_{(0,T)} \quad \text{for all } v \in H^{1/2}_{,0}(0,T)$$

Variational formulation: Find $u \in H^{1/2}_{0,}(0,T)$ such that

$$\langle \partial_t u, \mathcal{H}_T v \rangle_{(0,T)} = \langle f, \mathcal{H}_T v \rangle_{(0,T)} \quad \text{for all } v \in H^{1/2}_{0,}(0,T)$$

Theorem

$$\langle \partial_t u, \mathcal{H}_T u \rangle_{(0,T)} = \|u\|^2_{H^{1/2}_{0,}(0,T)} \quad \text{for all } u \in H^{1/2}_{0,}(0,T)$$

Corollary

$$\|u\|_{H^{1/2}_{0,}(0,T)} \leq \sup_{0 \neq v \in H^{1/2}_{,0}(0,T)} \frac{\langle \partial_t u, v \rangle_{(0,T)}}{\|v\|_{H^{1/2}_{,0}(0,T)}} \quad \text{for all } u \in H^{1/2}_{0,}(0,T)$$

Galerkin FEM: Find $u_h \in V_h := S_h^1(0,T) \cap H_{0,}^{1/2}(0,T)$ such that

$$\langle \partial_t u_h, \mathcal{H}_T v_h \rangle_{(0,T)} = \langle f, \mathcal{H}_T v_h \rangle_{(0,T)} \quad \text{for all } v_h \in V_h$$

Linear system

$$K_h \underline{u} = \underline{f}, \quad K_h = K_h^\top > 0$$

Numerical example: $u(t) = 8t^3 - 8t^2 + 2t$, $t \in (0,1)$

L	N	$\|u - u_h\|_{L^2}$	eoc	$\|u' - u_h'\|_{L^2}$	eoc	$\lambda_{\min}$	$\lambda_{\max}$	κ_2
0	2	1.949 –1		2.196		0.417	0.960	2.30
1	4	4.366 –2	2.20	1.152	0.93	0.284	1.117	3.93
2	8	1.018 –2	2.10	5.790 –1	0.99	0.169	1.128	6.68
3	16	2.440 –3	2.10	2.893 –1	1.00	0.091	1.133	12.38
4	32	5.964 –4	2.00	1.445 –1	1.00	0.047	1.134	23.89
5	64	1.474 –4	2.00	7.222 –2	1.00	0.024	1.134	46.96

Model problem ($\mu > 0$)

$$u'(t) + \mu u(t) = f(t) \quad \text{for } t \in (0,T), \quad u(0) = 0$$

Variational formulation: Find $u \in H^{1/2}_{0,}(0,T)$ such that

$$a(u,v) := \langle \partial_t u, v \rangle_{(0,T)} + \mu \langle u, v \rangle_{L^2(0,T)} = \langle f, v \rangle_{(0,T)} \quad \text{for all } v \in H^{1/2}_{,0}(0,T)$$

Theorem

$$a(u, \mathcal{H}_T u) = \langle \partial_t u, \mathcal{H}_T u \rangle_{(0,T)} + \mu \langle u, \mathcal{H}_T u \rangle_{L^2(0,T)} \geq \|u\|^2_{H^{1/2}_{0,}(0,T)}$$

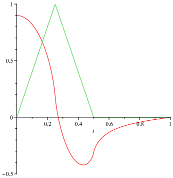
Galerkin FEM: Find $u_h \in V_h := S_h^1(0,T) \cap H^{1/2}_{0,}(0,T)$ such that

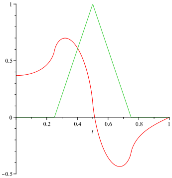
$$\langle \partial_t u_h, \mathcal{H}_T v_h \rangle_{(0,T)} + \langle u_h, \mathcal{H}_T v_h \rangle_{L^2(0,T)} = \langle f, \mathcal{H}_T v_h \rangle_{(0,T)} \quad \text{for all } v_h \in V_h$$

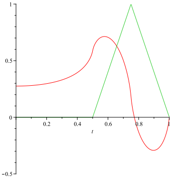
Numerical example ($\mu = 1$)

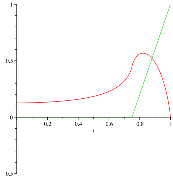
L	N	$\|u-u_h\|_{L^2}$	eoc	$\|u'-u_h'\|_{L^2}$	eoc	$\lambda_{\min}$	$\lambda_{\max}$	κ_2
0	2	1.947 –1		2.225		0.549	1.068	1.93
1	4	4.362 –2	2.20	1.155	0.95	0.390	1.126	2.84
2	8	1.017 –2	2.10	5.794 –1	1.00	0.236	1.131	4.68
3	16	2.439 –3	2.10	2.894 –1	1.00	1.289	1.133	8.59
4	32	5.963 –4	2.00	1.445 –1	1.00	0.067	1.134	16.56
5	64	1.473 –4	2.00	7.222 –2	1.00	0.034	1.134	32.58

Basis functions $\varphi_k, \mathcal{H}_T\varphi_k$









Find $u \in H^{1/2}_{0,}(0, T)$ such that

$$\langle \partial_t u, \mathcal{H}_T v \rangle_{(0,T)} + \mu \langle u, \mathcal{H}_T v \rangle_{L^2(0,T)} = \langle f, \mathcal{H}_T v \rangle_{(0,T)}$$

is satisfied for all $v \in H^{1/2}_{0,}(0, T)$.

Lemma: For $f \in [H^{1/2}_{,0}(0, T)]'$ we have

$$\|u\|^2_{L^2(0,T)} \leq \frac{T}{2} \sum_{k=0}^\infty \frac{\overline{f}_k^2}{\mu^2 + \frac{1}{T^2}(\frac{\pi}{2} + k\pi)^2},$$

where

$$\overline{f}_k := \frac{2}{T} \langle f, w_k \rangle_{(0,T)}, \quad w_k(t) := \cos\left(\left(\frac{\pi}{2} + k\pi \right) \frac{t}{T} \right).$$

Proof:

$$u(t) = \int_0^t e^{\mu(s-t)} f(s)\, ds = \sum_{k=0}^\infty \overline{f}_k \int_0^t e^{\mu s} \cos a_k s\, ds\, e^{-\mu t},$$

$$= \sum_{k=0}^\infty \frac{\overline{f}_k}{\mu^2 + a_k^2} \Big[a_k \sin a_k t + \mu \cos a_k t - \mu e^{-\mu t} \Big], \quad a_k = \frac{1}{T} \Big(\frac{\pi}{2} + k\pi \Big)$$

$$\|u\|^2_{L^2(0,T)} = \frac{T}{2} \sum_{k=0}^\infty \frac{\overline{f}_k^2}{\mu^2 + a_k^2} - \frac{1}{2}\mu \Big[1 + e^{-2\mu T} \Big] \left(\sum_{k=0}^\infty \frac{\overline{f}_k}{\mu^2 + a_k^2} \right)^2$$

$$\leq \frac{T}{2} \sum_{k=0}^\infty \frac{\overline{f}_k^2}{\mu^2 + a_k^2}$$

Lemma

$$\|u\|^2_{L^2(0,T)} \leq \frac{T}{2} \sum_{k=0}^\infty \frac{\overline{f}_k^2}{\mu^2 + \frac{1}{T^2}(\frac{\pi}{2} + k\pi)^2} \, ,$$

Remark

$$\|u\|^2_{L^2(0,T)} \leq \frac{T^3}{2} \sum_{k=0}^\infty \left(\frac{\pi}{2} + k\pi\right)^{-2} \overline{f}_k^2 \; = \; \|f\|^2_{[H^1_{,0}(0,T)]'}$$

For $f \in L^2(0,T)$:

$$\|u\|^2_{L^2(0,T)} \leq \frac{T}{2\mu^2} \sum_{k=0}^\infty \overline{f}_k^2 = \frac{1}{\mu^2} \|f\|^2_{L^2(0,T)}, \quad \text{i.e.} \quad \mu \, \|u\|_{L^2(0,T)} \leq \|f\|_{L^2(0,T)}.$$

Dirichlet boundary value problem for the heat equation

$$\begin{aligned} \alpha\partial_t u(x,t) - \Delta_x u(x,t) &= f(x,t) && \text{for } (x,t) \in Q := \Omega \times (0,T), \\ u(x,t) &= 0 && \text{for } (x,t) \in \Sigma := \Gamma \times (0,T), \\ u(x,0) &= 0 && \text{for } x \in \Omega. \end{aligned}$$

Find $u \in H^{1,1/2}_{0;0,} := L^2(0,T;H^1_0(\Omega)) \times H^{1/2}_{0,}(0,T;L^2(\Omega))$ such that

$$\int_0^T \int_\Omega \Big[\alpha\partial_t u(x,t)v(x,t) + \nabla_x u(x,t)\cdot\nabla_x v(x,t)\Big]dxdt = \int_0^T \int_\Omega f(x,t)v(x,t)dxdt$$

is satisfied for all $v \in H^{1,1/2}_{0;,0} := L^2(0,T;H^1_0(\Omega)) \times H^{1/2}_{,0}(0,T;L^2(\Omega))$.

Theorem

$$\frac{1}{2}\,\|u\|_{H^{1,1/2}_{0;0,}(Q)} \leq \sup_{0\neq v\in H^{1,1/2}_{0;,0}(Q)} \frac{a(u,v)}{\|v\|_{H^{1,1/2}_{0;,0}(Q)}}$$

Theorem The variational formulation implies an isomorphism

$$\mathcal{L}\colon\ H^{1,1/2}_{0;0,}(Q) \to [H^{1,1/2}_{0;\,,0}(Q)]',$$

satisfying

$$\|u\|_{H^{1,1/2}_{0;0,}(Q)} \leq 2\, \|\mathcal{L}u\|_{[H^{1,1/2}_{0;,0}(Q)]'} \quad \text{for all } u \in H^{1,1/2}_{0;0,}(Q)\,.$$

Proof: Write

$$u(x,t) = \sum_{i=1}^{\infty}\sum_{k=0}^{\infty} u_{i,k} v_k(t)\phi_i(x) = \sum_{i=1}^{\infty} U_i(t)\phi_i(x), \quad U_i(t) = \sum_{k=0}^{\infty} u_{i,k} v_k(t)$$

with norm

$$\|u\|^2_{H^{1,1/2}_{0;0,}(Q)} = \sum_{i=1}^{\infty}\Big[\|U_i\|^2_{H^{1/2}_{0,}(0,T)} + \mu_i \|U_i\|^2_{L^2(0,T)}\Big]$$

It remains to find $U_j \in H^{1/2}_{0,}(0,T)$ such that

$$\langle \partial_t U_j, V\rangle_{(0,T)} + \mu_j \langle U_j, V\rangle_{L^2(0,T)} = \langle f_j, V\rangle_{(0,T)}, \quad f_j(t) = \langle f, \phi_j\rangle_\Omega$$

is satisfied for all $V \in H^{1/2}_{,0}(0,T)$.

Then,

$$\begin{aligned}
\|U_j\|^2_{H^{1/2}_{0,}(0,T)} &= \langle \partial_t U_j, \mathcal{H}_T U_j \rangle_{(0,T)} \\
&\leq \langle \partial_t U_j, \mathcal{H}_T U_j \rangle_{(0,T)} + \mu_j \langle U_j, \mathcal{H}_T U_j \rangle_{L^2(0,T)} \\
&= \langle f_j, \mathcal{H}_T U_j \rangle_{(0,T)} \, .
\end{aligned}$$

For $M \in \mathbb{N}$ we define

$$u_M(x,t) = \sum_{j=1}^M U_j(t) \phi_j(x),$$

and we conclude

$$\begin{aligned}
\|u_M\|^2_{H^{1/2}_{0,}(0,T;L^2(\Omega))} &= \sum_{j=1}^M \|U_j\|^2_{H^{1/2}_{0,}(0,T)} \leq \sum_{j=1}^M \langle f_j, \mathcal{H}_T U_j \rangle_{(0,T)} \\
&= \int_0^T \int_\Omega f(x,t) \mathcal{H}_T \sum_{j=1}^M U_j(t) \phi_j(x) \, dx \, dt \\
&\leq \|f\|_{[H^{1,1/2}_{0;,0}(Q)]'} \|\mathcal{H}_T u_M\|_{H^{1,1/2}_{0;,0}(0,T)} = \|f\|_{[H^{1,1/2}_{0;,0}(Q)]'} \|u_M\|_{H^{1,1/2}_{0;0,}(0,T)} \, .
\end{aligned}$$

$$\partial_t U_i(t) + \mu U_i(t) = f_i(t) \quad \text{for } t \in (0,T), \quad U_i(0) = 0.$$

$$\|U_i\|^2_{L^2(0,T)} \leq \frac{T}{2} \sum_{k=0}^\infty \frac{\overline{f}^2_{i,k}}{\mu^2 + \frac{1}{T^2}(\frac{\pi}{2} + k\pi)^2}$$

Hence we obtain

$$\begin{aligned} \|u_M\|^2_{L^2(0,T;H^1_0(\Omega))} &= \sum_{i=1}^M \mu_i \|U_i\|^2_{L^2(0,T)} \leq \frac{T}{2} \sum_{i=1}^M \sum_{k=0}^\infty \frac{\mu_i}{\mu_i^2 + \frac{1}{T^2(\frac{\pi}{2}+k\pi)^2}} \overline{f}^2_{i,k} \\ &\leq T \sum_{i=1}^M \sum_{k=0}^\infty \frac{1}{\mu_i + \frac{1}{T(\frac{\pi}{2}+k\pi)}} \overline{f}^2_{i,k} \leq 2\, \|f\|^2_{[H^{1,1/2}_{0;,0}(Q)]'}, \end{aligned}$$

where we have used

$$\frac{a}{a^2+b^2} \leq \frac{a+b}{\frac{1}{2}(a+b)^2} = \frac{2}{a+b} \quad \text{for } 0 < a, b \in \mathbb{R}.$$

With this we have

$$\begin{aligned} \|u_M\|^2_{H^{1,1/2}_{0;0,}(Q)} &= \|u_M\|^2_{H^{1/2}_{0,}(0,T);L^2(Q)} + \|u_M\|^2_{L^2(0,T;H^1_0(\Omega))} \\ &\leq \|f\|_{[H^{1,1/2}_{0;,0}(Q)]'} \|u_M\|_{H^{1,1/2}_{0;0,}(0,T)} + 2\,\|f\|^2_{[H^{1,1/2}_{0;,0}(Q)]'}, \end{aligned}$$

and therefore

$$\|u_M\|_{H^{1,1/2}_{0;0,}(Q)} \leq 2\,\|f\|_{[H^{1,1/2}_{0;,0}(Q)]'}$$

follows for all $M \in \mathbb{N}$. Since the sequence $\{\|u_M\|_{H^{1,1/2}_{0;0,}(Q)}\}_{M\in\mathbb{N}}$ is increasing and uniformly bounded, the assertion follows for $M \to \infty$.

Galerkin FEM: Find $u_h \in V_h := Q_h^1(Q) \cap H_{0;0,}^{1,1/2}(Q)$ such that

$$\langle \partial_t u_h, \mathcal{H}_T v_h \rangle_Q + \langle \nabla_x u_h, \nabla_x \mathcal{H}_T v_h \rangle_{L^2(Q)} = \langle f, \mathcal{H}_T v_h \rangle_Q \quad \text{for all } v_h \in V_h$$

Numerical example

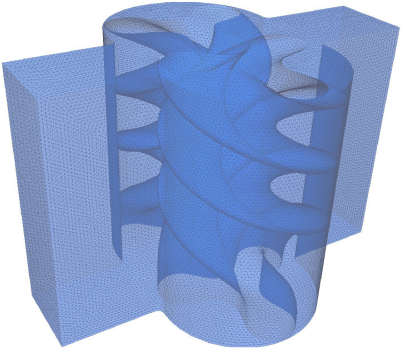
$$u(x,t) = \sin \frac{5\pi}{4} t \, \sin \pi x \quad \text{for } (x,t) \in (0,1) \times (0,2), \quad \alpha = 1$$

N_x	N_t	dof	h_x	h_t	$\|u - u_h\|_{L^2}$	eoc	$\|u - u_h\|_{H^1}$	eoc
2	2	2	0.50000	1.0000	0.91080		4.48436	
4	4	12	0.25000	0.5000	0.15774	2.50	1.89083	1.20
8	8	56	0.12500	0.2500	0.02936	2.40	0.84239	1.20
16	16	240	0.06250	0.1250	0.00689	2.10	0.41495	1.00
32	32	992	0.03125	0.0625	0.00169	2.00	0.20679	1.00

- coercive space–time FEM
  - modified Hilbert transformation for arbitrary meshes
  - stable pairs of ansatz and test functions
  - adaptivity simultaneously in space and time
  - parallel iterative solution and preconditioners
- space–time BEM in trace spaces
  - mapping properties of boundary integral operators
  - Calderon system of boundary integral equations
- space–time BEM/FEM coupling
- applications
- ...

Adaptive space–time discretizations [M. Neumüller]

- moving geomtries (Lobe pump)



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